General shift invariant systems and Gabor frames over irregular lattices in local fields of positive characteristic

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Abstract

In this paper, we have discussed some necessary and sufficient conditions for dual frames formed by shift invariant systems which are also Bessel sequences. Further, explicit expression for Walnut operator associated with window function of Gabor frames in local fields of positive characteristics is given. Also properties of Gabor frames over irregular lattices in local fields of positive characteristics are discussed.
Keywords: Shift invariant system, Dual frame, Gabor frame, Local fields of positive characteristic.

1 Introduction

Frame in an inner product space is a generalization of basis of a vector space to sets that may be linearly dependent. A frame provides a redundant, stable way of representing a signal.

In 1946, Gabor [10] first proposed the study of systems $\{E_{mb}T_{na}g(\cdot) = e^{2\pi imb\cdot}g(\cdot - na) : m, n \in \mathbb{Z}\}$ with $ab = 1$. The systems $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ are potential tools for the decomposition and handling of signals, like speech and music, that have well-defined frequencies over short intervals and it change with time. For $g = \chi_{[0,1]}$, the system $(E_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$.

Given $a, b > 0$ and $g \in L^2(\mathbb{R})$, we say that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ generates a Gabor frame, or Weyl Heisenberg (WH) frame for $L^2(\mathbb{R})$ if the family $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ consisting of modulates and translates of a single function g, forms a frame for $L^2(\mathbb{R})$. The function g is then said to be a **mother wavelet**. Here, a, b are the frame parameters, with a being the shift parameter, and b the modulation parameter.
A.J.E.M. Janssen presented formulations of the conditions for duality of Weyl-Heisenberg systems in the time domain, the frequency domain, the time-frequency domain, and, for rational time-frequency sampling factors, the Zak transform domain, both for the one dimensional time-continuous case and the time- discrete case.

This paper consists of four sections. In the second section basic results related to local field of positive characteristics are given. In the third section conditions equivalent for dual frames formed by shift invariant Bessel sequences and Gabor frames are discussed. Further, Walnut representation for the frame operator is given and shift invariant frames are characterized. In the fourth section some properties of Gabor frames over irregular lattices are given.

2 Preliminaries and basic results related to LFPC

A collection of elements $\{f_j : j \in J\}$ in a Hilbert space H is called a frame if there exist constants A and B, $0 < A \leq B < \infty$, such that

$$A\|f\|^2 \leq \sum_{j\in J} |\langle f, f_j\rangle|^2 \leq B\|f\|^2,$$

for all $f \in H$. A and B are called lower and upper frame bounds, respectively. We say that the frame is a **tight frame** if $A = B$, and if $A = B = 1$, then the frame is said to be **normalized tight frame, or Parseval frame**.

If $\{f_j : j \in J\}$ is a frame, then frame operator $S : H \to H$ is defined as $Sf = \sum_{j\in J}\langle f, f_j\rangle f_j$. S is a bounded, invertible operator, and $f = \sum_{j\in J}\langle S^{-1}f, f_j\rangle f_j = \sum_{j\in J}\langle f, S^{-\frac{1}{2}}f_j\rangle S^{-\frac{1}{2}}f_j$. It can be seen easily that $\{f_j : j \in J\}$ is a normalized tight frame if and only if $S = I$.

Suppose $\{f_j\}_{j\in J}$ is a frame for Hilbert space H. Then the sequence $\{g_j\}_{j\in J}$ is said to be **dual** to $\{f_j\}_{j\in J}$ if

$$f = \Sigma_{j\in J}\langle f, g_j\rangle f_j$$

for all $f \in H$.

A field $\mathbb{K}$ equipped with a topology is called local field if both the additive group $\mathbb{K}^+$ and multiplicative group $\mathbb{K}^*$ of $\mathbb{K}$ are locally compact abelian groups. For example, any field with discrete topology is a local field.

Let $\mathbb{K}$ be a local field with the ring of integers $\mathfrak{D} = \{x \in \mathbb{K} : |x| \leq 1\}$. Let dx denotes the Haar measure on $\mathbb{K}^+$. The field $\mathbb{K}$ is non-trivial, locally compact, totally disconnected and complete topological field endowed with non-Archimedean norm $|.| : \mathbb{K} \to \mathbb{R}^+$ satisfying

(i) $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$.

(ii) $|xy| = |x|.|y|$, for all $x, y \in \mathbb{K}$.

(iii) $|x + y| \leq max\{|x|, |y|\}$, for all $x, y \in \mathbb{K}$.

Let $\mathfrak{B} = \{x \in \mathbb{K} : |x| < 1\}$ be a prime ideal of the ring of integers $\mathfrak{D}$. The quotient space $\frac{\mathfrak{D}}{\mathfrak{B}}$ is isomorphic to finite field $GF(q)$,where $q = p^c$, for some prime p and $c \in \mathbb{N}$. As $\mathbb{K}$ is totally disconnected and $\mathfrak{B}$ is both prime and principal ideal, it follows that there exists a prime element $\mathfrak{p}$ of $\mathbb{K}$ such that $\mathfrak{B} = \langle\mathfrak{p}\rangle = \mathfrak{p}\mathfrak{D}$. Let $\mathfrak{D}^* = \{x \in \mathbb{K} : |x| = 1\}$. Then $\mathfrak{D}^*$ is the group of units in $\mathbb{K}^*$. We can write $0 \neq x$ as $x = \mathfrak{p}^n y,\ y \in \mathfrak{D}^*$. Let $\mathfrak{U} = \{u_m : m = 0, 1, \cdots, q-1\}$ be a fixed set of all coset representative of $\mathfrak{B}$ in $\mathfrak{D}$. Then every element in $\mathfrak{B}$ is written uniquely as $x = \sum_{l=k}^{\infty} c_l\mathfrak{p}^l$, where $c_l \in \mathfrak{U}, c_k \neq 0$. Define $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \{x \in \mathbb{K} : |x| < q^{-k}\}$, for all $k \in \mathbb{Z}$. Then each $\mathfrak{B}^k$ is a compact, open subgroup of $\mathbb{K}^+$.

Suppose χ is a character on $\mathbb{K}^+$, which is trivial on $\mathfrak{D}$ but non trivial on $\mathfrak{B}^{-1}$, then χ is constant on cosets of $\mathfrak{D}$. If $y \in \mathbb{K}^+$, then $\chi_y(x) = \chi(x, y)$. Let χ_u be any character on $\mathbb{K}^+$. Since $\mathfrak{D}$ is a subgroup of $\mathbb{K}^+$, it follows that restriction $\chi_{u|\mathfrak{D}}$ is character on $\mathfrak{D}$. Since $\chi_{u|\mathfrak{D}}$ is a character on $\mathfrak{D}$, $\chi_u = \chi_v$ iff $u - v \in \mathfrak{D}$.

Theorem 2.1. *[21] Let* $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$ *be a complete list of distinct coset representative of* $\mathfrak{D}$ *in* $\mathbb{K}^+$*. Then* $\{\chi_{u(n)|\mathfrak{D}} \equiv \chi_{u(n)}\}_{n\in\mathbb{N}_0}$ *is a complete list of distinct characters on* $\mathfrak{D}$*. Moreover, it is a complete orthonormal system on* $L^2(\mathfrak{D})$*.*

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Theorem 2.2. *[21] For* $n \in \mathbb{N}_0$*, let* $u(n)$ *be defined as above. Then*

(i) $u(n) = 0$ *if and only if* $n = 0$*. If* $k \geq 1$*, then we have* $|u(n)| = q^k$ *if and only if* $q^{k-1} \leq n < q^k$*.*

(ii) $\{u(k) : k \in \mathbb{N}_0\} = \{-u(k) : k \in \mathbb{N}_0\}$*.*

(iii) For a fixed $l \in \mathbb{N}_0$*, we have* $\{u(l) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$*.*

Theorem 2.3. *[13] For all* $l, k \in \mathbb{N}_0$*, we have* $\chi_{u(k)}(u(l)) = 1$*.*

A function f defined on $\mathbb{K}$ is said to be integral periodic if $f(x+u(l)) = f(x)$ for all $l \in \mathbb{N}_0$, $x \in \mathbb{K}$.

For $f \in L^2(\mathbb{K})$, the **Fourier transform** of f is defined as

$$\hat{f}(\xi) = \int_{\mathbb{K}} f(x)\overline{\chi(\xi x)}dx. \tag{2.1}$$

The Fourier transform is an unitary mapping of $L^2(\mathbb{K})$ onto itself.

Notation: For $a \in \mathbb{K}$, we define $\mathbb{B}_a = \{x \in \mathbb{K} : |x| \leq |a|\}$. An orthonormal basis for $L^2(\mathbb{B}_a)$ is given by $\{|a|^{-\frac{1}{2}}\chi_{u(n)a^{-1}}\}_{n\in\mathbb{N}_0}$.

We define operations of **modulation** and **translation** for $f \in L^2(\mathbb{K})$ by

$$E_{u(m)b}f(x) = \chi(u(m)b)f(x) \tag{2.2}$$

and

$$T_{u(n)a}f(x) = f(x - u(n)a) \tag{2.3}$$

respectively.

A sequence $\{y_n\}_{n\in\mathbb{N}_0}$ in $\mathbb{K}$ is said to be $|\delta|$**-separated**, if there exists a $\delta \in \mathbb{K} - \{0\}$ such that $|y_n - y_m| \geq |\delta|$ for all $n \neq m$.

3 General Shift invariant system on LFPC

Let $\{g_m\}_{m\in\mathbb{N}_0}$ be a collection of functions in $L^2(\mathbb{K})$ and $a \in \mathbb{K} - \{0\}$ be a given parameter. The shift-invariant system generated by $\{g_m\}_{m\in\mathbb{N}_0}$ and a is the collection of functions $\{g_{nm}\}_{n,m\in\mathbb{N}_0}$ where $g_{mn} = g_m(. - u(n)a)$.

Lemma 3.1. *Assume that shift invariant systems* $\{g_{nm}\}_{n,m\in\mathbb{N}_0}$ *and* $\{h_{nm}\}_{n,m\in\mathbb{N}_0}$ *are Bessel sequences and let* $e, f \in L^2(\mathbb{K})$. *Then the function* $\rho(e,f) : \mathbb{K} \longrightarrow \mathbb{C}, \rho(e,f)(x) = \Sigma_{m,n\in\mathbb{N}_0}\langle T_x e, g_{nm}\rangle\langle h_{nm}, T_x f\rangle$ *is continuous and has period* a. *Its Fourier series in* $L^2(\mathbb{B}_a)$ *is*

$$\rho(e,f)(x) = \Sigma_{k\in\mathbb{N}_0} c_k \chi_{u(k)a^{-1}}(x) \tag{3.1}$$

where $c_k = |a|^{-1}\int_{\mathbb{K}} \hat{e}(\nu)\overline{\hat{f}(\nu + u(k)a^{-1})}\Sigma_{m\in\mathbb{N}_0}\overline{\hat{g_m}(\nu)\hat{h}_m.(\nu + u(k)a^{-1})}d\nu$.

Proof The function $\rho(e,f)$ is well defined by the assumption that $\{g_{nm}\}_{m,n\in\mathbb{N}_0}$ and $\{h_{nm}\}_{m,n\in\mathbb{N}_0}$ are Bessel sequences. In fact, the series defining $\rho(e,f)(x)$ converges absolutely for all $x \in \mathbb{K}$ as

$$\begin{aligned}\Sigma_{m,n\in\mathbb{N}_0}|\langle T_x e, g_{nm}\rangle\langle h_{nm}, T_x f\rangle| &\le (\Sigma_{m,n\in\mathbb{N}_0}|\langle T_x e, g_{nm}\rangle|^2)^{\frac{1}{2}}(\Sigma_{m,n\in\mathbb{N}_0}|\langle h_{nm}, T_x f\rangle|^2)^{\frac{1}{2}}\\ &\le B\|e\|.\|f\|\end{aligned}$$

where B is a common Bessel bound for both the sequences $\{g_{nm}\}_{m,n\in\mathbb{N}_0}$ and $\{h_{nm}\}_{m,n\in\mathbb{N}_0}$ in $L^2(\mathbb{K})$. The continuity follows from a similar argument. Given $x, x_o \in \mathbb{K}$,we have

$$\begin{aligned}|\rho(e,f)(x) - \rho(e,f)(x_o)| &= |\Sigma_{m,n\in\mathbb{N}_0}(\langle T_x e, g_{nm}\rangle\langle h_{nm}, T_x f\rangle - \langle T_{x_o} e, g_{nm}\rangle\langle h_{nm}, T_{x_o} f\rangle)|\\ &\le \Sigma_{m,n\in\mathbb{N}_0}|\langle T_x e - T_{x_o} e, g_{nm}\rangle\langle h_{nm}, T_x f\rangle - \langle T_{x_o} e, g_{nm}\rangle\langle h_{nm}, T_{x_o} f - T_x f\rangle|\\ &\le (\Sigma_{m,n\in\mathbb{N}_0}|\langle T_x e - T_{x_o} e, g_{nm}\rangle|^2)^{\frac{1}{2}})(\Sigma_{m,n\in\mathbb{N}_0}|\langle h_{nm}, T_x f\rangle|^2)^{\frac{1}{2}})+\\ &(\Sigma_{m,n\in\mathbb{N}_0}|\langle T_{x_o} e, g_{nm}\rangle|^2)^{\frac{1}{2}}(\Sigma_{m,n\in\mathbb{N}_0}|\langle h_{nm}, T_x f - T_{x_o} f\rangle|^2)^{\frac{1}{2}}\\ &\le B\|f\|.\|T_x e - T_{x_o} e\| + B\|e\|.\|T_x f - T_{x_o} f\|.\end{aligned}$$

The last expression converges to zero as $x \longrightarrow x_o$. The periodicity of $\rho(e,f)$ follows from the structure of the shift-invariant systems $\{g_{nm}\}_{m,n\in\mathbb{N}_0}$ and $\{h_{nm}\}_{m,n\in\mathbb{N}_0}$. Given $l \in \mathbb{N}_o$, note that

$$\begin{aligned}\rho(e,f)(x + u(l)a) &= \Sigma_{m,n\in\mathbb{N}_0}\langle T_{x+u(l)a} e, g_{nm}\rangle\langle h_{nm}, T_{x+u(l)a} f\rangle\\ &= \Sigma_{m,n\in\mathbb{N}_0}\langle T_x T_{u(l)a} e, g_{nm}\rangle\langle h_{nm}, T_x T_{u(l)a} f\rangle\\ &= \Sigma_{m,n\in\mathbb{N}_0}\langle T_{u(l)a} T_x e, g_{nm}\rangle\langle h_{nm}, T_{u(l)a} T_x f\rangle\\ &= \Sigma_{m,n\in\mathbb{N}_0}\langle T_x e, T_{-u(l)a} g_{nm}\rangle\langle T_{-u(l)a} h_{nm}, T_x f\rangle\\ &= \Sigma_{m,n\in\mathbb{N}_0}\langle T_x e, g_{nm}\rangle\langle h_{nm}, T_x f\rangle.\end{aligned}$$

For the computation of the Fourier coefficients, we first assume that e, f are continuous functions with compact support, this will justify all interchanges of sums and integrals.
The coefficients in the Fourier expansion with respect to the orthonormal basis

$\{|a|^{-\frac{1}{2}}\chi_{u(k)a^{-1}}\}_{k\in\mathbb{N}_o}$ of $L^2(\mathbb{B}_a)$ are given by

$$\begin{aligned} c_k &= |a|^{-1}\int_{\mathbb{B}_a} \rho(e,f)(x)\overline{\chi_{u(k)a^{-1}}(x)}dx \\ &= |a|^{-1}\Sigma_{m,n\in\mathbb{N}_o}\int_{\mathbb{B}_a} \langle T_x e, g_{nm}\rangle\langle h_{nm}, T_x f\rangle\overline{\chi_{u(k)a^{-1}}(x)}dx \\ &= |a|^{-1}\Sigma_{m\in\mathbb{N}_o}\int_{\mathbb{K}} \langle T_x e, g_m\rangle\overline{\langle T_x f, h_m\rangle\chi_{u(k)a^{-1}}(x)}dx. \end{aligned}$$

Now for any arbitrary function $\phi \in L^2(\mathbb{K})$,

$$\begin{aligned} \langle T_x e, \phi\rangle &= \langle \mathbb{F}T_x e, \mathbb{F}\phi\rangle \\ &= \langle E_{-x}\hat{e}, \hat{\phi}\rangle \\ &= \int_{\mathbb{K}} \hat{e}(\nu)\overline{\hat{\phi}(\nu)}\chi_{-x}(\nu)d\nu \\ &= \mathbb{F}(\hat{e}\overline{\hat{\phi}})(x). \end{aligned}$$

where $\mathbb{F}$ denotes the Fourier transform.

Thus,

$$\begin{aligned} \langle T_x f, h_m\rangle\chi_{u(k)a^{-1}}(x) &= \mathbb{F}(\hat{f}\overline{\hat{h}}_m)(x)\chi_{u(k)a^{-1}}(x) \\ &= E_{u(k)a^{-1}}(\mathbb{F}(\hat{f}\overline{\hat{h}}_m)(x) \\ &= \mathbb{F}(T_{-u(k)a^{-1}}(\hat{f}\overline{\hat{h}}_m))(x) \end{aligned}$$

Using the above expression and the fact that Fourier transform is unitary , we get

$$\begin{aligned} c_k &= |a|^{-1}\Sigma_{m\in\mathbb{N}_0}\int_{\mathbb{K}} \langle T_x e, g_m\rangle\overline{\langle T_x f, h_m\rangle\chi_{u(k)a^{-1}}(x)}dx \\ &= |a|^{-1}\Sigma_{m\in\mathbb{N}_o}\int_{\mathbb{K}} \mathbb{F}(\hat{e}\overline{\hat{g}}_m)(x)\overline{\mathbb{F}(T_{-u(k)a^{-1}}(\hat{f}\overline{\hat{h}}_m(x))}dx \\ &= |a|^{-1}\int_{\mathbb{K}} \hat{e}(\nu)\overline{\hat{f}(\nu+u(k)a^{-1})}(\Sigma_{m\in\mathbb{N}_o}\overline{\hat{g}_m(\nu)}\hat{h}_m(\nu+u(k)a^{-1}))d\nu \end{aligned}$$

This proves the result in case of $e, f \in C_c(\mathbb{K})$. Since, $C_c(\mathbb{K})$ is dense in $L^2(\mathbb{K})$, result also holds for any $e, f \in L^2(\mathbb{K})$.

Observation If the conditions of the above lemma holds, then we have

$$\Sigma_m|\hat{g}_m(\nu)|^2 \leq |a|B_g, \Sigma_m|\hat{h}_m(\nu)|^2 \leq |a|B_h \tag{3.2}$$

where B_g and B_h are Bessel bounds for the sequences $\{g_{nm}\}_{n,m\in\mathbb{N}_0}$ and $\{h_{nm}\}_{n,m\in\mathbb{N}_0}$ respectively.

Theorem 3.1. *Two shift-invariant systems $\{g_{nm}\}_{n,m\in\mathbb{N}_0}$ and $\{h_{nm}\}_{n,m\in\mathbb{N}_0}$ which form Bessel sequences are dual frames if and only if*

$$\Sigma_{m\in\mathbb{N}_0}\overline{\hat{g}_m(\nu)}\hat{h}_m(\nu+u(k)a^{-1}) = |a|\delta_{k,0} \tag{3.3}$$

for each $k \in \mathbb{N}_0$ and a.e. ν.

Proof We know that Bessel sequences $\{g_{nm}\}_{n,m\in\mathbb{N}_o}, \{h_{nm}\}_{n,m\in\mathbb{N}_o}$ are dual frames if and only if $\langle e,f\rangle = \Sigma_{m,n\in\mathbb{N}_o}\langle e,g_{nm}\rangle\langle h_{nm},f\rangle$ for all $e,f\in L^2(\mathbb{K})$[25]. If we assume that $\{g_{nm}\}_{n,m\in\mathbb{N}_o}, \{h_{nm}\}_{n,m\in\mathbb{N}_o}$ are dual frames, it follows from the above identity that

$$\begin{aligned}\rho(e,f)(x) &= \Sigma_{m,n\in\mathbb{N}_o}\langle T_x e,g_{nm}\rangle\langle h_{nm},T_x f\rangle = \langle T_x e,T_x f\rangle\\ &= \langle e,f\rangle,\end{aligned}$$

for all $e,f\in L^2(\mathbb{K})$. Hence, the function $\rho(e,f)$ and the constant function $\langle e,f\rangle$ have the same Fourier coefficients in $L^2(\mathbb{B}_a)$ i.e,

$$\begin{aligned}|a|^{-1}\int_{\mathbb{K}}\hat{e}(\nu)\overline{\hat{f}(\nu+u(k)a^{-1})}(\Sigma_{m\in\mathbb{N}_o}\overline{\hat{g}_m(\nu)}\hat{h}_m(\nu+u(k)a^{-1}))d\nu &= \langle e,f\rangle\delta_{k,o} = \langle\hat{e},\hat{f}\rangle\delta_{k,o}\\ &= \delta_{k,o}\int_{\mathbb{K}}\hat{e}(\nu)\overline{\hat{f}(\nu)}d\nu.\end{aligned}$$

Since this holds for all $e,f\in L^2(\mathbb{K})$, we have

$$\Sigma_{m\in\mathbb{N}_o}\overline{\hat{g}_m(\nu)}\hat{h}_m(\nu+u(k)a^{-1}) = |a|\delta_{k,0}$$

for a.e ν. The opposite implication is similar; assuming that given condition holds, it follows that functions $\rho(e,f)$ and the constant function $\langle e,f\rangle$ have the same Fourier coefficients. Since, from previous lemma when $k=0$, we have $c_k = \langle\hat{e},\hat{f}\rangle = \langle e,f\rangle$ and when $k\neq 0, c_k = 0$. As $\rho(e,f)$ is continuous, it follows that $\rho(e,f)(x) = \langle e,f\rangle$ for all $x\in\mathbb{K}$. Thus for $x=0$, we have

$$\begin{aligned}\rho(e,f)(0) &= \langle e,f\rangle\\ &= \Sigma_{m,n\in\mathbb{N}_o}\langle e,g_{nm}\rangle\langle h_{nm},f\rangle\end{aligned}$$

This completes the proof.

Theorem 3.2. *Let $g,h\in L^2(\mathbb{K})$ and $a,b\in\mathbb{K}-\{0\}$ be given. Then, if the two Gabor systems $\{E_{u(m)b}T_{u(n)a}g\}_{m,n\in\mathbb{N}_0}$ and $\{E_{u(m)b}T_{u(n)a}h\}_{m,n\in\mathbb{N}_0}$ are Bessel sequences, they are dual frames if and only if*

$$\langle h, E_{u(m)a^{-1}}T_{u(n)b^{-1}}g\rangle = 0 \qquad (3.4)$$

for all $(m,n)\neq(0,0)$ and $\langle h,g\rangle = |ab|$.

Proof The Bessel sequences $\{E_{u(m)b}T_{u(n)a}g\}_{m,n\in\mathbb{N}_0}$ and $\{E_{u(m)b}T_{u(n)a}h\}_{m,n\in\mathbb{N}_0}$ are dual frames if and only if the shift-invariant systems $\{T_{u(n)a}E_{u(m)b}g\}_{m,n\in\mathbb{N}_0}$ and $\{T_{u(n)a}E_{u(m)b}h\}_{m,n\in\mathbb{N}_0}$ are dual frames. The generator for the two latter systems are

$$g_m = E_{u(m)b}g \quad \text{and} \quad h_m = E_{u(m)b}h.$$

By previous theorem they generate dual frame if and only if

$$\Sigma_{m\in\mathbb{N}_o}\overline{\hat{g}_m(\nu)}\hat{h}_m(\nu+u(k)a^{-1}) = |a|\delta_{k,0},$$

for a.e. ν and each $k \in \mathbb{N}_0$. As $\hat{g}_m = T_{-u(m)b}\hat{g}$, we can express this condition in terms of the coefficients in the Fourier expansion with respect to $\{|b|^{-\frac{1}{2}}\chi_{u(m)b^{-1}}\}$ for the $b-$ periodic functions $\phi_k(\nu) = \Sigma_{m\in\mathbb{N}_o}\overline{\hat{g}(\nu - u(m)b)}\hat{h}(\nu + a^{-1}u(k) - u(m)b), k \in \mathbb{N}_o$. In fact the above statement is equivalent to all coefficients for $\phi_k, k \neq 0$ being zero and the coefficients for ϕ_0 being zero for $m \neq 0$ and equal to $|a|$ for $m = 0$. The $nth-$ coefficient for the function ϕ_k in the Fourier expansion with respect to $\{|b|^{-\frac{1}{2}}\chi_{u(m)b^{-1}}\}$ is

$$\begin{aligned}|b|^{-\frac{1}{2}}\int_{\mathbb{B}_b}\Sigma_{m\in\mathbb{N}_0}\overline{\hat{g}(\nu - u(m)b)}\hat{h}(\nu + a^{-1}u(k) - u(m)b)\chi_{-u(n)}(b^{-1}\nu)d\nu &= |b|^{-\frac{1}{2}}\langle T_{-a^{-1}u(k)}\hat{h}, E_{u(n)b^{-1}}\hat{g}\rangle\\ &= |b|^{-\frac{1}{2}}\langle \hat{h}, T_{u(k)a^{-1}}E_{u(n)b^{-1}}\hat{g}\rangle\\ &= |b|^{-\frac{1}{2}}\langle \mathbb{F}h, \mathbb{F}(E_{-u(k)a^{-1}}T_{(u(n)b^{-1}}g)\rangle\\ &= |b|^{-\frac{1}{2}}\langle h, E_{-u(k)a^{-1}}T_{(u(n)b^{-1}}g\rangle.\end{aligned}$$

Lemma 3.2. *Assume that the system $g_{nm}, (n,m) \in \mathbb{N}_0^2$ is a Bessel sequence with Bessel bound B_g. Then for a.e. $\nu \in \mathbb{K}, H_{g,\nu} : l^2(\mathbb{N}_0) \longrightarrow l^2(\mathbb{N}_0)$ defined as*

$$H_{g,\nu}(\beta) = \{\Sigma_m \hat{g}_m(\nu - u(k)a^{-1})\beta_m\}_{k\in\mathbb{N}_0}$$

where $\beta = \{\beta_m\}_{m\in\mathbb{N}_0} \in l^2(\mathbb{N}_0)$, defines a bounded linear operator on $l^2(\mathbb{N}_0)$ with operator norm $\leq (|a|B_g)^{\frac{1}{2}}$. Explicitly, we have for a.e.ν

$$\Sigma_k|\Sigma_m \hat{g}_m(\nu - u(k)a^{-1})\beta_m|^2 \leq |a|B_g\|\beta\|^2 \tag{3.5}$$

where $\|\beta\|^2 = \Sigma_m|\beta_m|^2$ gives the norm of $\beta \in l^2(\mathbb{N}_0)$.

Proof Suppose $\underline{\alpha} = \{\alpha_{nm}\}_{n,m\in\mathbb{N}_0}$ in $l^2(\mathbb{N}_0^2)$ is such that $\alpha_{nm} \neq 0$ for only finitely many n, m and let $\alpha_m(\nu) = \Sigma_n\alpha_{nm}\overline{\chi_{u(n)a}(\nu)}, \quad \nu \in \mathbb{K}, m \in \mathbb{N}_0$. Note that α_m is a^{-1} periodic for each $m \in \mathbb{N}_0$. We have,

$$\begin{aligned}\int_{\mathbb{B}_{a^{-1}}}\Sigma_k|\Sigma_m\alpha_m(\nu)\hat{g}_m(\nu - u(k)a^{-1})|^2 d\nu &= \Sigma_k\int_{\mathbb{B}_{a^{-1}}+u(k)a^{-1}}|\Sigma_m\alpha_m(\nu)\hat{g}_m(\nu)|^2 d\nu\\ &= \int_{\mathbb{K}}|\Sigma_{n,m}\alpha_{nm}\overline{\chi_{u(n)a}(\nu)}\hat{g}_m(\nu)|^2 d\nu\\ &= \|\Sigma_{n,m}\alpha_{nm}g_{nm}\|^2\end{aligned}$$

Now, $T_g^* : l^2(\mathbb{N}_0^2) \longrightarrow L^2(\mathbb{K})$ defined by $T_g^*(\underline{\alpha}) = \Sigma_{n,m}\alpha_{nm}g_{nm}$ is a bounded linear operator with operator norm $\leq B_g^{\frac{1}{2}}$.
Also, $|\underline{\alpha}|^2 = \Sigma_{n,m}|\alpha_{nm}|^2 = |a|\int_{\mathbb{B}_{a^{-1}}}\Sigma_m|\alpha_m(\nu)|^2 d\nu$.

$$\int_{\mathbb{B}_{a^{-1}}}\Sigma_k|\Sigma_m\alpha_m(\nu)\hat{g}_m(\nu - u(k)a^{-1})|^2 d\nu \leq B_g\|\underline{\alpha}\|^2 = |a|B_g\int_{\mathbb{B}_{a^{-1}}}\Sigma_m|\alpha_m(\nu)|^2 d\nu. \tag{3.6}$$

Next fix $\underline{\beta} = \{\beta_m\}_{m\in\mathbb{N}_0} \in l^2(\mathbb{N}_0)$ for only finitely many $m \in \mathbb{N}_0$ and choose $\alpha_m(\nu) = \beta_m\phi(\nu)$; $\phi(\nu) = \Sigma_n\phi_n\overline{\chi_{u(n)a}(\nu)}$ with $\phi_n \neq 0$ for finitely many $n \in \mathbb{N}_0$. Then, inequality (3.6) gives

$$\int_{\mathbb{B}_{a^{-1}}} |\phi(\nu)|^2\Sigma_k|\Sigma_m\hat{g}_m(\nu - u(k)a^{-1})\beta_m|^2 d\nu \leq |a|B_g\|\underline{\beta}\|^2 \int_{\mathbb{B}_{a^{-1}}} |\phi(\nu)|^2 d\nu. \tag{3.7}$$

Setting $\phi(\gamma) = \overline{\chi_{u(n)a}(\gamma)}, n \in \mathbb{N}_0$ we see that

$$\Sigma_k|\Sigma_m\hat{g}_m(\nu - u(k)a^{-1})\beta_m|^2 \leq |a|B_g\|\underline{\beta}\|^2 \quad \text{a.e.}\nu \in \mathbb{B}_{a^{-1}} \tag{3.8}$$

where the null set in (3.8) may depend on $\underline{\beta}$.

Since the above ineqaulity holds on a dense set of $l^2(\mathbb{N}_0)$, it holds for all sequences in $l^2(\mathbb{N}_0)$.

Theorem 3.3. *Assume that the systems $g_{nm}, (n,m) \in \mathbb{N}_0^2$ is a Bessel sequence with Bessel bound B_g and let $f \in L^2(\mathbb{K})$. Then , we have*

$$\widehat{S_g f}(\nu) = |a|^{-1}\Sigma_k d_k(\nu)\hat{f}(\nu - u(k)a^{-1}), k \in \mathbb{N}_0, \tag{3.9}$$

where $d_k(\nu) = \Sigma_m\hat{g}_m(\nu)\overline{\hat{g}_m(\nu - u(k)a^{-1})}$, $k \in \mathbb{N}_0$.

Proof We have by Lemma 3.1 and Lemma 3.2

$$\Sigma_k|d_k(\nu)|^2 = \Sigma_k|\Sigma_m\hat{g}_m(\nu)\overline{\hat{g}_m(\nu - u(k)a^{-1})}|^2 \tag{3.10}$$

$$\leq |a|B_g\Sigma_m|\hat{g}_m(\nu)|^2 \tag{3.11}$$

$$\leq (|a|B_g)^2. \tag{3.12}$$

Since $f \in L^2(\mathbb{K})$, $\Sigma_k|\hat{f}(\nu - u(k)a^{-1})|^2 < \infty$ for a.e. $\nu \in L^2(\mathbb{K})$, so that right hand side of equation (3.9) is a.e. well- defined as an absolutely convergent series.

Now let $h \in C_c$. We shall show that

$$\langle S_g f, h\rangle = |a|^{-1}\int_{\mathbb{K}}(\Sigma_k d_k(\nu)\hat{f}(\nu - u(k)a^{-1}))\hat{h}(\nu)d\nu \tag{3.13}$$

From this and the above, the result follows by density of C_c in $L^2(\mathbb{K})$ and Parseval's theorem. To show (3.13), we observe that by (3.12) and the Hölder's inequality

$$\Sigma_k\int_{\mathbb{K}} |d_k(\nu)||\hat{f}(\nu - u(k)a^{-1})||\hat{h}(\nu)|d\nu \leq |a|B_g\int_{\mathbb{K}}(\Sigma_k|\hat{f}(\nu - u(k)a^{-1}))|^2)^{\frac{1}{2}}|\hat{h}(\nu)|d\nu \tag{3.14}$$

The right hand side of (3.14) is easily seen to be finite , whence the function $\rho(f,h)$ of Lemma (3.1), with $h_m = g_m, m \in \mathbb{N}_0$, has an absolutely convergent Fourier series, therefore,

$$\langle S_g f, h\rangle = \rho(f,h)(0) = \Sigma_k c_k \tag{3.15}$$

$$= \Sigma_k |a|^{-1} \int_{\mathbb{K}} \hat{f}(\nu)\overline{\hat{h}(\nu + u(k)a^{-1})}\Sigma_m \overline{\hat{g}_m(\nu)}\hat{g}_m(\nu + u(k)a^{-1})d\nu \tag{3.16}$$

$$= \Sigma_k |a|^{-1} \int_{\mathbb{K}} d_k(\nu)\hat{f}(\nu - u(k)a^{-1})\overline{\hat{h}(\nu)}d\nu \tag{3.17}$$

By inequality (3.14), the series and integral in (3.17) may be interchanged and we arrive at (3.13). This completes the proof.

Theorem 3.4. *Let $g_m \in L^2(\mathbb{K}), m \in \mathbb{N}_0$ and let $A \geq 0, B < \infty$. Then,*

$$A\|f\|^2 \leq \Sigma_{n,m}|\langle f, g_{nm}\rangle|^2 \leq B\|f\|^2, f \in L^2(\mathbb{K}) \tag{3.18}$$

iff

$$|a|A \leq \|H_{g,\nu}\|\|H^*_{g,\nu}\| \leq |a|B \quad a.e.\nu \in \mathbb{K}. \tag{3.19}$$

Proof When the right hand inequality of (3.18) holds, we have that $\|H_{g,\nu}\|\|H^*_{g,\nu}\| \leq |a|B$ by Lemma (3.2), since $H_{g,\nu}$ and $H^*_{g,\nu}$ have same operator norm.
Next let $f \in C_c$ with f compactly supported, and let $m \in \mathbb{N}_0$. Then

$$\Sigma_k \hat{g}_m(\nu - u(k)a^{-1})\overline{\hat{f}(\nu - u(k)a^{-1})} \equiv \Sigma_n c_{nm}\chi_{u(n)a}(\nu) \tag{3.20}$$

with

$$c_{nm} = |a|^{-1}\langle g_{nm}, f\rangle, \quad n \in \mathbb{N}_0. \tag{3.21}$$

Thus,

$$\int_{\mathbb{B}_{a^{-1}}} |\Sigma_k \hat{g}_m(\nu - u(k)a^{-1})\overline{\hat{f}(\nu - u(k)a^{-1})}|^2 d\nu = |a|\Sigma_n|\langle f, g_{nm}\rangle|^2. \tag{3.22}$$

For $\nu \in \mathbb{K}$, let

$$\underline{\hat{f}}(\nu) = (\hat{f}(\nu - u(k)a^{-1}))_{k\in\mathbb{N}_0}. \tag{3.23}$$

Note that

$$\|f\|^2 = \int_{\mathbb{B}_{a^{-1}}} \|\underline{\hat{f}}(\nu)\|^2 d\nu. \tag{3.24}$$

Then from right hand inequality of (3.18) and (3.22), we see that

$$\int_{\mathbb{B}_{a^{-1}}} \|H_{g,\nu}(\underline{\hat{f}}(\nu))\|^2 d\nu = \int_{\mathbb{B}_{a^{-1}}} \Sigma_m|\Sigma_k \hat{g}_m(\nu - u(k)a^{-1})\hat{f}^*(\nu - u(k)a^{-1})|^2 d\nu \tag{3.25}$$

$$= |a|\Sigma_{nm}|\langle f, g_{nm}\rangle|^2 \tag{3.26}$$

Now we now need to show that $\|H_{g,\nu}(\underline{\beta})\|^2 \geq |a|A\|\underline{\beta}\|^2$ for $\underline{\beta} \in l^2(\mathbb{N}_0)$. Let $\hat{\phi} \in C_c$ be supported on $\mathbb{B}_{a^{-1}}$ and let $\underline{\beta} \in l^2(\mathbb{N}_0)$ with $\beta_k \neq 0$ for only finitely many $k \in \mathbb{N}_0$. For $\nu \in \mathbb{K}$, define $\hat{f}(\nu) \equiv \beta_k \hat{\phi}(\nu + u(k)a^{-1})$, where $k \in \mathbb{N}_0$ is such that $\nu + u(k)a^{-1} \in \mathbb{B}_{a^{-1}}$.Then

$$\underline{\hat{f}}(\nu) = \hat{\phi}(\nu)\underline{\beta}, \quad \text{for } \nu \in \mathbb{B}_{a^{-1}} \tag{3.27}$$

Therefore, we get from (3.22), the first inequality of (3.18) and

$$\int_{\mathbb{B}_{a^{-1}}} |\hat{\phi}(\nu)|^2 \|H^*_{g,\nu}(\underline{\beta})\|^2 d\nu = |a|\Sigma_{nm}|\langle f, g_{nm}\rangle|^2 \geq |a|A\|f\|^2$$

$$= |a|A \int_{\mathbb{B}_{a^{-1}}} \|\underline{\hat{f}}(\nu)\|^2 d\nu = |a|A\|\underline{\beta}\|^2 \int_{\mathbb{B}_{a^{-1}}} |\hat{\phi}(\nu)|^2 d\nu.$$

By varying $\hat{\phi}$ over all elements of C_c supported in $\mathbb{B}_{a^{-1}}$, we get

$$\|H^*_{g,\nu}(\underline{\beta})\|^2 \geq |a|A\|\underline{\beta}\|^2, \quad \text{for a.e } \ \nu \in \mathbb{B}_{a^{-1}}, \tag{3.28}$$

where the null set involved in (3.28) may depend on $\underline{\beta}$.
Since the above ineqaulity holds on a dense set of $l^2(\mathbb{N}_0)$, it holds for all sequences in $l^2(\mathbb{N}_0)$.

For the converse part, let $\hat{f}$ be boundedly supported. Then (3.24) and (3.26) show that the quantity $|a|\Sigma_{m,n}|\langle f, g_{nm}\rangle|^2$ lies between $|a|A\|f\|^2$ and $|a|B\|f\|^2$.

4 Gabor frames over irregular lattices in LFPC

Proposition 4.1. *Let $h, g \in L^2(\mathbb{K})$, then the series*

$$\Sigma_{n\in\mathbb{N}_0} h(x - u(n)a)\overline{g(x - u(n)a - y_k)} \tag{4.1}$$

converges absolutely a.e. for all $a, y_k \in \mathbb{K} - \{0\}$,$k \in \mathbb{N}_0$.

Proposition 4.2. *Let $f \in L^2(\mathbb{K})$. Suppose that $b, y_n \in \mathbb{K} - \{0\}$ for all $n \in \mathbb{N}_0$ and $(E_{u(m)b}T_{y_n}g)_{m,n\in\mathbb{N}_0}$ is tight WH-frame for $L^2(\mathbb{K})$, then*

$$\Sigma_m\Sigma_n|\langle f, E_{u(m)b}T_{y_n}g\rangle|^2 = \|f\|^2 = F_1(f) + F_2(f), \tag{4.2}$$

where

$$F_1(f) = |b|^{-1}\int_{\mathbb{K}} |f(x)|^2\Sigma_n|g(x - y_n)|^2 dx, \tag{4.3}$$

and

$$F_2(f) = |b|^{-1}\Sigma_{k\geq 1}\int_{\mathbb{K}} f(x - b^{-1}u(k))\overline{f(x)}\Sigma_{n\in\mathbb{N}_0} g(x - y_n) \tag{4.4}$$

$$\overline{g(x - y_n - b^{-1}u(k))}dx. \tag{4.5}$$

Proposition 4.3. *Let $h, g \in L^2(\mathbb{K})$ and $b \in \mathbb{K} - \{0\}$.*

(i) $h \perp E_{u(m)b} g$ *for all* $m \neq 0$ *if and only if there is a constant* C *such that*

$$\Sigma_{n \in \mathbb{N}_0} h(x - b^{-1}u(n))\overline{g(x - b^{-1}u(n))} = C \quad a.e. \tag{4.6}$$

(ii) If $y_n \neq 0$*, then* $h \perp E_{u(m)b} T_{y_n} g$*, for all* $m \in \mathbb{N}_0$ *if and only if*

$$\Sigma_{k \in \mathbb{N}_0} h(x - b^{-1}u(k))\overline{g(x - b^{-1}u(k) - y_n)} = 0 \quad a.e. \tag{4.7}$$

Theorem 4.1. *Let* $y_n, n \in \mathbb{N}_0$ *be a sequence in* $\mathbb{K}$ *such that it is* $|\delta|$*-separated and* $\{\mathbb{B}_b + y_n\}_{n \in \mathbb{N}_0}$ *is a partition of* $\mathbb{K}$*, for some* $\delta \in \mathbb{K} - \{0\}$*. Then*

$$G(x) = \Sigma_{n \in \mathbb{N}_0} |g(x - u(n)b^{-1})|^2 = |\delta| \tag{4.8}$$

a.e. and

$$G_m(x) = \Sigma_{n \in \mathbb{N}_0} g(x - u(n)a)\overline{g(x - b^{-1}u(n) - y_m)} = 0 \quad a.e. \tag{4.9}$$

for $y_m \neq 0$ *iff* $g \perp E_{u(n)b} T_{y_m} g$*, for all* $n, y_m \neq 0$ *and* $\|g\|^2 = |\delta b|$*.*

Theorem 4.2. *Let* $g \in L^2(\mathbb{K}), a \in \mathbb{K} - \{0\}$*. Then* $g \perp E_{u(n)a^{-1}} T_{y_m} g$*, for all* $n, y_m \neq 0$ *iff* $(E_{u(n)a^{-1}} T_{y_m} g)_{m,n \in \mathbb{N}_0}$ *is an orthogonal sequence in* $L^2(\mathbb{K})$*.*

Theorem 4.3. *If* $g \in L^2(\mathbb{K})$ *and* $y_n, b \in \mathbb{K} - \{0\}$ *for all* $n \in \mathbb{N}_0$ *are such that* $(E_{u(m)b} T_{y_n} g)_{m,n \in \mathbb{N}_0}$ *is a frame for* $L^2(\mathbb{K})$ *with frame bounds* $A, B > 0$*, then we must have*

$$A|b| \leq \Sigma_{k \in \mathbb{N}_0} |g(x - y_k)|^2 \leq B|b| \tag{4.10}$$

a.e. In particular, g *is bounded.*

The proofs for above results are similar to results proved in case of Gabor frames over regular lattices.

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