FRACTAL DIMENSION AND FRACTIONAL CALCULUS OF NON-STATIONARY ZIPPER α-FRACTAL FUNCTIONS

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The fractal interpolation functions with appropriate iterated function systems (IFSs) provide a method to perturb and approximate a continuous function on a compact interval I. This method produces a class of functions f^{α}, named as α-fractal functions. As essential parameters of the IFS, the scaling factor α has important consequences in the properties of the function f^{α}. In this talk, we discuss the α-fractal functions corresponding to the non-stationary zipper IFS. Here, we present a method to calculate an upper bound of the box and Hausdorff dimension of the proposed interpolant. Also, we provide an upper bound of the graph of the fractional integral of the proposed interpolant.

Fractal dimension and fractional calculus of non-stationary zipper α-fractal functions

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Overview

- Introduction and motivation
- Construction of fractal functions .
- Non-stationary zipper α-fractal functions.
- Fractal dimension and fractional calculus.

Introduction

(X, d)-Complete metric space.

$H(X) = \{A \subset X : A \neq \phi,\ A$ is compact$\}$.
The Hausdorff metric h on $H(X)$ is defined as

$$h(A, B) = \max\{d(A, B), d(B, A)\},$$

$d(A, B) = \max \min d(x, y),\ x \in A, y \in B$.

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Iterated Function System (IFS): $\{X; w_n, n = 1, 2, \ldots, N-1\}$, w_n are continuous maps on X.

Contractive IFS: The IFS is called hyperbolic if w_n are contraction maps with contractive factors $|\alpha_n| < 1$.

Attractor

The Hutchinson map on $H(X)$ is defined as $W(A) = \cup_{n=1}^{N-1} w_n(A)$ [1].

W is a contraction map on $(H(X), h)$ with contractive factor $s = \max\{|s_n| : n \in J\},\ J = \{1, 2, \ldots, N-1\}$.

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By Banach's Fixed Point Theorem, $\lim\limits_{m\to\infty} W_m(A) = G$.

The unique fixed point is known as **Attractor** or **Deterministic Fractal** of the IFS.

Examples: Sierpiński triangle, Cantor set, Koch curve.

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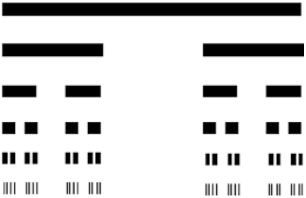
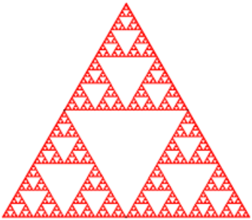
Examples of Fractals



(a) Cantor Set



(b) Sierpinski Triangle

(c) Fern

(d) Cauliflower

Construction of Fractal Interpolation Functions (FIFs)

- Consider increasing data points: $\{(x_i, y_i), i = 1, 2, \ldots, N\}$. Let $L_i : I = [a, b] \mapsto I_i = [x_i, x_{i+1}], i \in \{1, 2, \ldots, N-1\}$ with $L_i(x_1) = x_i,\ L_i(x_N) = x_{i+1}$.

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- Let $K = I \times \mathbb{R}$ and $w_i(x, y) = (L_i(x), F_i(x, y))$, where $F_i : K \mapsto \mathbb{R}$ satisfy $F_i(x_1, y_1) = y_i, F_i(x_N, y_N) = y_{i+1}$ and

$$|F_i(x, y) - F_i(x, y')| \leq \alpha_i |y - y'|,\ \forall\, (x, y), (x, y') \in K,\ \ 0 \leq \alpha_i < 1.$$

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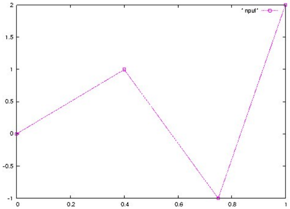
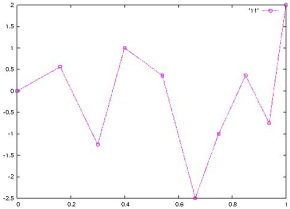
Theorem (Barnsley, 1986)

The IFS $\mathcal{I} = \{K; w_i : i = 1, 2, \dots, N\}$ admits a unique attractor G. Further, G is the graph of a continuous function $f : I \mapsto \mathbb{R}$ which obeys $f(x_i) = y_i$ for $i = 1, 2, \dots, N$.
The previous function is called a FIF

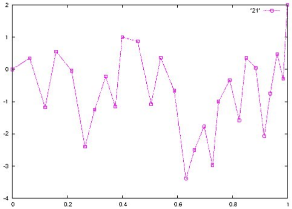
FIF for the data $\{(0,0),(0.4,1),(0.75,-1),(1,2)\}$, with $\alpha_i = 0.8$



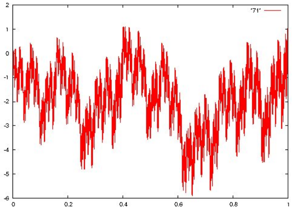
(e) Linear Interpolation



(f) FIF after One Iteration



(g) FIF after Two Iterations



(h) FIF after Seven Iterations

Sequence of Zipper IFSs

- Let w_i be non-surjective maps on a complete metric space X. Then the system $\mathcal{I} = \{X; w_i : i \in \mathbb{N}_N\}$ is called a zipper [2] with vertices $(v_0, v_1, \dots, v_N)$ and signature $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{0,1\}^n$ if for any $i = 1, 2, \dots, n$,

$$w_i(v_0) = v_{i-1+\varepsilon_i}, \quad w_i(v_N) = v_{i-\varepsilon_i}.$$

[2] V. V. Aseev, On the regularity of self-similar zippers, Materials, 24-30, (2002)

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$$w_i(v_0) = v_{i-1+\varepsilon_i}, \quad w_i(v_N) = v_{i-\varepsilon_i}.$$

- Let $P_{i,\epsilon} = a_i x + bi, F_{i,k}(x,y) = \alpha_{i,k}(x)y + q_{i,k}(x)$. For $i \in \mathbb{N}_{N-1}$, we define $W_{i,k} : K \to I_i \times \mathbb{R}$ by

$$W_{i,k}(x,y) = \big(P_{i,\boldsymbol{\epsilon}}(x), F_{i,k}(x,y)\big),$$

which forms a sequence of zipper IFSs $\mathcal{I}_k := \big\{K; W_{i,k} : i \in \mathbb{N}_{N-1}\big\}$.

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Sequence of Transformations and Trajectories

Consider a sequence of transformations $\{T_i\}_{i\in\mathbb{N}}$, $T_i : X \to X$.
For $W_k = \{w_{1,k}, w_{2,k}, \ldots, w_{n_k,k}\}$, consider the sequence of set valued maps

$$W_k(A) = \bigcup_{i=1}^{n_k} w_{i,k}(A), A \in H(X). \tag{1.1}$$

[3]M.F. Barnsley, M.F., J.E. Hutchinson, O. Stenfow, V-variable fractals: fractals with partial self similarity. Adv. Math., 2008

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Forward and Backward Trajectories: The forward and backward trajectories are defined as

$$\Phi_k := T_k \circ T_{k-1} \circ \ldots T_1 \text{ and } \Psi_k := T_1 \circ T_2 \circ \ldots T_k.$$

[3]

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Convergence of Trajectories

Theorem (Levin, Dyn, Viswanathan)

Let $\{W_k\}_{k\in\mathbb{N}}$ *be a family of set-valued maps as described in* (1.1)*, where* $W_k = \{w_{i,k} : i \in \mathbb{N}_{n_k}\}$ *of contractions on* (X, d)*. Assume that*

(i) *there exxists a nonempty closed invariant set* $\mathcal{P} \subset X$ *for* $w_{i,k}, i \in \mathbb{N}_{n_k}, k \in \mathbb{N}$ *and*

(ii) $\sum_{k=1}^{\infty} \prod_{j=1}^{k} \text{Lip}(W_j) < \infty.$

Then the backward trajectories $\{\Psi_k(A)\}$ *converges for any initial* $A \subseteq \mathcal{P}$ *to a unique attractor* $G \subseteq \mathcal{P}$*.*

Non-stationary α-fractal functions

Notation: $A := \{\alpha_k\}_{k\in\mathbb{N}}$ and $s := \{s_k\}_{k\in\mathbb{N}}$. Let $\mathcal{C}_f(I) := \big\{g \in \mathcal{C}(I) : g(x_1) = f(x_1), g(x_N) = f(x_N)\big\}$. It is obvious that $\mathcal{C}_f(I)$ is a complete metric space. For $k \in \mathbb{N}$, we define a sequence of RB operators $T^{\alpha_k}_{s_k,\boldsymbol{\epsilon}} : \mathcal{C}_f(I) \to \mathcal{C}_f(I)$ by

$$(T^{\alpha_k}_{s_k,\boldsymbol{\epsilon}} g)(x) = F_{i,k}(Q_{i,\boldsymbol{\epsilon}}(x), g(Q_{i,\boldsymbol{\epsilon}}(x)) \quad \forall\, x \in I_i,\ i \in \mathbb{N}_{N-1},$$

where $Q_{i,\boldsymbol{\epsilon}}(x) := P^{-1}_{i,\boldsymbol{\epsilon}}(x)$.

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Proposition

Let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of Lipschitz maps on a complete metric space X. If there exists $x_* \in X$ such that the sequence $\{d(x_*, T_k(x_*))\}$ is bounded, and $\sum_{k=1}^{\infty} \prod_{i=1}^{k} c_i < \infty$ then the sequence $\{\Psi_k(x)\}$ converges for all $x \in X$ to a unique limit $\overline{x}$.

Theorem

Consider the sequence of operators $\{T^{\alpha_k}_{s_k,\epsilon}\}$ *on* $\mathcal{C}_f(I)$*. Then for every* $g \in \mathcal{C}_f(I)$ *the sequence* $\{T^{\alpha_1}_{s_1,\epsilon} \circ T^{\alpha_2}_{s_2,\epsilon} \circ \cdots \circ T^{\alpha_k}_{s_k,\epsilon} g\}$ *converges to a map* $f^A_{s,\epsilon}$ *of* $\mathcal{C}_f(I)$.

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Proof: Step 1: Construct the backward trajectories.
Step 2: Define the RB operator using it.
Step 3: Use the convergence result and find a bound of $\|T^{\alpha_k}_{s_k,\epsilon} f - f\|_\infty$.
Then apply previous theorem.

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- Answers: Known for stationary cases.

1. Box-dimension of linear FIFs: $dim_B(G) = s \in (1, 2), \sum_{i=1}^{N} a_i^{s-1}|d_i| = 1$, when $\sum_{i=1}^{N} |d_i| > 1$, partition points are not collinear (Barnsley-Elton-Hardin-Massopust, SIAM J.M.A, 1989).

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  2. Hausdorff dimension of an affine FIF: $\min\{2, l\} \leq dim_H(G) \leq u,$ where l, u are the positive solutions of $\sum_{n=1}^{N} t_n^l = 1, \sum_{n=1}^{N} s_n^u = 1$, when $t_1.t_N \leq \min(a_1, a_N)\left(\sum_{n=1}^{N} t_n^l\right)$ (Barnsley, Const. Approx., 1986).

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  3. A particular class of FIF: $dim_H(G) = s$, where s is the unique solution of $\sum_{i=1}^{k} |\mu|\lambda_i^{s-1} = 1$ (Gibert-Massopust, JMAA, 1992).

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4. Bilinear FIFs: $dim_B(G) = 1 + \log \frac{\gamma}{N}, \gamma = \sum_{n=1}^{N} \frac{s_n + s_{n-1}}{2} > 1$ (Barnsley-Massopust, JAT, 2015).

Box and Hausdorff dimension

- Let F be a nonempty bounded subset of $\mathbb{R}^n$ and let $N_\delta(F)$ denote the smallest number of sets of diameter less than or equal to δ which covers F.
- The lower and upper box-counting dimension of F is defined as

$$\underline{\dim}_B(F) = \lim \inf_{\delta \to 0^+} \frac{N_\delta(F)}{-\log \delta}, \ \overline{\dim}_B(F) = \lim \sup_{\delta \to 0^+} \frac{N_\delta(F)}{-\log \delta}.$$

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- The s-dimensional Hausdorff measure is defined as

$$H^s(F) = \lim_{\delta\to 0} \inf\{\sum_{i=1}^{\infty} |U_i|^s : F \subset \cup_{i=1}^{\infty} U_i, |U_i| < \delta\}$$

- The Hausdorff dimension of F is defined by $\dim_H(F) = \inf\{s \geq 0 : H^s(F) = 0\}$ and for any bounded subset F of $\mathbb{R}^n$,

$$\dim_H(F) \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F).$$

4

[4] K. Falconer, Fractal Geometry, 2nd ed., Mathematical Foundations and

Computation of Fractal Dimension

For Hölder continuous (HC) functions f with exponent σ, let us define σth Hölder seminorm as

$$[f]_\sigma = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\sigma}.$$

Consider the Hölder space
$\mathcal{H}^\sigma(I) := \{g : I \to \mathbb{R} :$ g is HC with exponent $\sigma\}$.

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Theorem

Let f and $\alpha_{i,k}$ be HC with exponent σ_1 and σ_2 respectively for every $k \in \mathbb{N}$. Let s_k be HC with exponent σ_3 satisfying $s_k(x_i) = f(x_i)$ for $i \in \{1, N\},\ k \in \mathbb{N}$. If $\max\left\{\|\alpha_k\|_\sigma, \frac{\|\alpha_k\|_\infty}{(\min\{|a_i|\})^\sigma}\right\} < 1,\ \forall\ k \in \mathbb{N}$, then

$$1 \leq \dim_H \left(\mathit{Graph}(f^A_{s,\epsilon})\right) \leq \underline{\dim}_B\left(\mathit{Graph}(f^A_{s,\epsilon})\right) \leq 2 - \sigma,$$

where $\sigma = \min\{\sigma_1, \sigma_2, \sigma_3\}$ and $\|\alpha_k\|_\sigma = \max\{\|\alpha_{i,k}\|_\sigma : i \in \mathbb{N}_{N-1}\}$.

Let

$$\mathcal{BV}(I) := \{f : I \to \mathbb{R}; \quad f \ \text{ is of bounded variation on } I\}.$$

Then $(\mathcal{BV}(I), \|.\|_{\mathcal{BV}})$ is complete, where $\|f\|_{\mathcal{BV}} := |f(t_0)| + V(f, I)$.

Theorem (Liang, 2010)

If $f \in \mathcal{C}(I) \cap \mathcal{BV}(I)$, then

$$\dim_H(Graph(f)) = \dim_B(Graph(f)) = 1.$$

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Theorem

Let $f \in \mathcal{BV}(I)$. Suppose that $\triangle$ is a partition of I, $s_k \in \mathcal{BV}(I)$ satisfying $s_k(x_1) = f(x_1)$, $s_k(x_N) = f(x_N)$, and $\alpha_{i,k}$ $(i \in \mathbb{N}_{N-1}, k \in \mathbb{N})$ are functions in $\mathcal{BV}(I)$ with

$$\|\alpha_k\|_{\mathcal{BV}} := \max\{\|\alpha_{i,k}\|_{\mathcal{BV}} : i \in \mathbb{N}_{N-1}\} < \frac{1}{2(N-1)}, \ \forall \ k \in \mathbb{N}.$$

Then, $f^A_{s,\epsilon} \in \mathcal{BV}(I)$ and $\dim_H\left(Gf(f^A_{s,\epsilon})\right) = \dim_B\left(Gf(f^A_{s,\epsilon})\right) = 1$.

Let $0 < \alpha < 1$. The Riemann-Liouville fractional integral of order of an integrable function $g : [a, b] \to \mathbb{R}$ is

$$_a\mathfrak{J}^{\alpha} g(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)\ \mathrm{d}t.$$

In 2007, Liang proved that

$$\dim_B \left(\text{Graph}({}_a\mathfrak{J}^{\alpha} f)\right) = 1, \quad \text{whenever } f \in \mathcal{BV}(I).$$

Fractional Calculus

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Recently, using covering method, he obtained the following:

$$\dim_B \left(\text{Graph}(_a\mathfrak{J}^{\alpha} f)\right) = 1, \quad \text{whenever } \dim_B \left(\text{Graph}(f)\right) = 1.$$

Apart from these works, Ruan et al. established a linear relationship between the order of fractional integral and box dimension of two linear FIFs.

Box dimension of fractional integral

Theorem

Let $f \in \mathcal{BV}(I)$ *and consider an increasing partition of* I. *Let* $s_k \in \mathcal{BV}(I)$ *be such that* $s_k(x_1) = f(x_1),\ s_k(x_N) = f(x_N)$, *and* $\alpha_{i,k}$ $(i \in \mathbb{N}_{N-1},\ k \in \mathbb{N})$ *are functions in* $\mathcal{BV}(I)$ *with* $\|\alpha_k\|_{\mathcal{BV}} < \frac{1}{2(N-1)}\ \forall\ \ k \in \mathbb{N}$. *Then,*

$$\dim_H \left(Gf({}_a\mathfrak{J}^{\alpha} f^{A}_{s,\boldsymbol{\epsilon}}) \right) = \dim_B \left(Gf({}_a\mathfrak{J}^{\alpha} f^{A}_{s,\boldsymbol{\epsilon}}) \right) = 1.$$

Bounds of the dimension

Theorem

Let f and $\alpha_{i,k}$ be Hölder continuous with exponent σ_1 and σ_2 respectively for every $k \in \mathbb{N}$. Let s_k be Hölder continuous with exponent σ_3 satisfying $s_k(x_i) = f(x_i)$ for $i \in \{1, N\},\ k \in \mathbb{N}$. If $\max\left\{\|\alpha_k\|_\sigma, \frac{\|\alpha_k\|_\infty}{(\min\{|a_i|\})^\sigma}\right\} < 1 \ \ \forall \ \ k \in \mathbb{N}$, then

$$1 \leq \dim_H \left(Gf({}_a\mathfrak{J}^\alpha f^A_{s,\epsilon})\right) \leq \underline{\dim}_B\left(Gf({}_a\mathfrak{J}^\alpha f^A_{s,\epsilon})\right) \leq \overline{\dim}_B\left(Gf({}_a\mathfrak{J}^\alpha f^A_{s,\epsilon})\right) \leq \min\{2-\alpha, 2-\sigma\},$$

where $\sigma = \min\{\sigma_1, \sigma_2, \sigma_3\}$.

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[5] S. Jha, S. Verma, A.K.B. Chand, Non-stationary zipper α-fractal functions and associated fractal operator. Fract. Calc. Appl. Anal., 2022

Idea of the proof

- Step 1: $f_{s,\epsilon}^{A} \in \mathcal{H}^{\sigma}(I)$.
- Step 2: Let $0 < a \leq x < x+h \leq b$. We have

$$
\begin{aligned}
{}_a\mathfrak{I}^{\alpha} f_{s,\epsilon}^{A}(x+h) -_a \mathfrak{I}^{\alpha} f_{s,\epsilon}^{A}(x) =& \frac{1}{\Gamma(\alpha)} \int_a^{x+h} (x+h-t)^{\alpha-1} f_{s,\epsilon}^{A}(t)dt \\
& - \frac{1}{\Gamma(\alpha)} \int_a^{x} (x^{\rho+1} - t^{\rho+1})^{\alpha-1} f_{s,\epsilon}^{A}(t)dt. \\
=& I_1 + I_2,
\end{aligned}
$$

- Step 3: Find bound of I_1, I_2.
- Find bound of $N_{\delta}(Gf({}_a\mathfrak{I}^{\alpha} f_{s,\epsilon}^{A}))$.

Proof continues

- $N_\delta(Gf({}_a\mathfrak{I}^\alpha f^A_{s,\boldsymbol{\epsilon}})) \leq 2\left\lceil\frac{b-a}{\delta}\right\rceil + \sum_{i=1}^{\left\lceil\frac{b-a}{\delta}\right\rceil} \frac{2M}{\Gamma(\alpha+1)}\delta^{\alpha-1}$.
- Consequently,

$$\overline{\dim}_B\big(Graph({}_a\mathfrak{I}^\alpha f^A_{s,\boldsymbol{\epsilon}})\big) = \overline{\lim_{\delta\to 0}} \frac{\log N_\delta(Graph({}_a\mathfrak{I}^\alpha f^A_{s,\boldsymbol{\epsilon}}))}{-\log\delta} \leq 2-\alpha.$$

- To show $\overline{\dim}_B\big(Graph({}_a\mathfrak{I}^\alpha f^A_{s,\boldsymbol{\epsilon}})\big) \leq 2-\sigma$, find

$${}_a\mathfrak{I}^\alpha f^A_{s,\boldsymbol{\epsilon}}(x+h) -_a \mathfrak{I}^\alpha f^A_{s,\boldsymbol{\epsilon}}(x).$$

Note

In [6], for a linear FIF g, which is determined by

$\{L_i(x), F_i(x,y)\}_{i=1}^{N-1}$, where $L_i(x) = a_i x + b_i$ and $F_i(x,y) = d_i y + q_i(x)$

are such that $\sum_{i=1}^{N-1} |d_i| > 1$, $\dim_B(Gf(g)) = D(\{a_i, d_i\})$ and $\sum_{i=1}^{N-1} |d_i| a_i^{D(\{a_i,b_i\})-1} = 1$, it is shown that

$$\dim_B(Gf({}_a\mathfrak{I}^\alpha g)) = \dim_B(Gf(g)) - \alpha,$$

for any $0 < \alpha < D(\{a_i, d_i\}) - 1$.

[6] H.-J. Ruan, Su, W.-Y., Yao, K.: Box dimension and fractional integral of linear fractal interpolation functions. J. Approx. Theory, 2009

Thank you for your attention