Numerical and analytical investigation for solutions of fractional Oskolkov-Benjamin-Bona-Mahony-Burgers equation describing propagation of long surface waves

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Abstract

In this article, a novel meshless numerical scheme to solve the time-fractional Oskolkov-Benjamin-Bona-Mahony-Burgers equation has been proposed. This equation able to describe many nonlinear phenomena such as analysis of the long-wavelength surface waves in liquids, acoustic-gravity waves in compressible fluids and hydromagnetic waves in cold plasma. The proposed numerical scheme is based on finite difference and Kansa-radial basis function collocation approach. Firstly, the finite difference scheme has been employed to discretize the time-fractional derivative and subsequently, the Kansa method is utilized to discretize the spatial derivatives. The stability and convergence of the proposed numerical scheme are also elucidated in this article. Also, the Kudryashov technique has been used to obtain the soliton solutions for comparison with the numerical results. Finally, numerical simulations are performed to confirm the applicability and accuracy of the proposed scheme.

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Outline

Introduction

- Fractional partial differential equations (FPDEs) are widely used to describe numerous complex real life problems in many fields of science and engineering, such as fluid dynamics, reaction-diffusion, wave propagation, plasma physics, and many other physical and biological processes [1].
- It is significant to find new analytical and numerical solutions of these equations for understanding the physical phenomenon.
- In recent years, radial basis functions (RBFs) are rigorously utilized for solving PDEs.
- In 1990, a meshless method, also known as Kansa method, suggested by Kansa [2] as a tool for solving PDEs utilizing collocation and RBFs, particularly the multiquadric (MQ).

- The time-fractional Oskolkov-Benjamin-Bona-Mahony-Burgers (OBBMB) equation [3] as

$$D_t^{\beta} u - D_t^{\beta} u_{xx} - \alpha u_{xx} + \gamma u_x + \theta u u_x = 0, \; 0 < \beta \leq 1, \tag{1}$$

  where $u(x, t)$ denotes the fluid velocity in the horizontal direction x and α, γ, θ are nonzero real parameters with $\alpha > 0$. Here, $\beta(0 < \beta \leq 1)$ indicates the order of Caputo fractional derivative.

- The Eq.(1) able to describe many nonlinear phenomena such as analysis of the long-wavelength surface waves in liquids, acoustic-gravity waves in compressible fluids, and hydromagnetic waves in cold plasma.

Methodology

Algorithm of Kudryashov technique

- A nonlinear FPDE in the polynomial form given by

$$E(u, u_x, u_{xx}, ..., D_t^{\beta} u, D_t^{\beta} u_x, ...) = 0, \quad 0 < \beta \leq 1. \tag{2}$$

- Utilizing the FCT [4], $u(x,t) = \Phi(\zeta), \quad \zeta = \nu\left(x - \frac{\kappa t^{\beta}}{\Gamma(\beta+1)}\right)$, the FPDE (2) is transformed to the ODE as

$$E(\Phi, \nu\Phi', \nu^2\Phi'', ..., -\nu\kappa\Phi', -\nu^2\kappa\Phi', ...) = 0, \tag{3}$$

- Suppose that the exact solution of Eq.(3) can be expressed in the form

$$\Phi(\zeta) = \sum_{i=0}^{M} a_i Q^i(\zeta), \quad Q(\zeta) = \frac{1}{1+\exp(\zeta)}, \tag{4}$$

where the function $Q(\zeta)$ satisfies the first order ODE $Q_{\zeta} = Q^2 - Q$.

- Substituting Eq.(4) in (3), collecting all terms with the same powers of Q and equating to zero, a system of algebraic equations acquired and by solving them the exact solutions of Eq.(2) can be obtained.

Consider a finite set of scattered node points $\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N\} \subset \mathbb{R}^d$ and a function $u : \Omega \to \mathbb{R}$. According to Kansa method of interpolation utilizing RBFs, the interpolant of u can be written in the following form [5]:

$$(\mathcal{P}u)(\mathbf{x}) = \sum_{j=1}^{N} \eta_j \phi(\|\mathbf{x} - \mathbf{x}_j\|) + \sum_{k=1}^{l} \eta_{N+k} p_k(\mathbf{x}), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \qquad (5)$$

where $l = \binom{m+d-1}{m-1}$, $\|.\|$ is the Euclidean norm and $\{p_k(\mathbf{x})\}_{k=1}^{l}$ represents the basis of Π_{m-1}^d.

Now, we impose the interpolation and regularization conditions to determine the values of unknown coefficients $\{\eta_j\}_{j=1}^{N+l}$ as follows:

$$\begin{cases} (\mathcal{P}u)(\mathbf{x}_j) = u(\mathbf{x}_j), \quad j = 1, 2, ..., N, \\ \displaystyle\sum_{j=1}^{N} \eta_j p_k(\mathbf{x}_j) = 0, \quad\quad k = 1, 2, ..., l. \end{cases} \tag{6}$$

The numerical simulations in the present work have been carried out with MQ-RBF, which is defined as follows:

$$\phi(\|\mathbf{x} - \mathbf{x}_j\|) = \phi(r_j) = \sqrt{r_j^2 + c^2}, \tag{7}$$

where c denotes shape parameter and $r_j = \|\mathbf{x} - \mathbf{x}_j\|$ is the Euclidean norm.

Analytical solutions

Using the FCT,

$$u(x,t) = \Psi(\zeta),\ \zeta = k_1\left(x - \frac{c_1 t^{\beta}}{\Gamma(\beta+1)}\right),$$

Eq.(1) can be transformed to the nonlinear ODE as

$$-c_1\Psi' + k_1^2 c_1 \Psi''' - k_1\alpha\Psi'' + \gamma\Psi' + \theta\Psi\Psi' = 0. \tag{8}$$

Integrating the above equation with respect to ζ yields

$$-c_1\Psi + k_1^2 c_1 \Psi'' - k_1\alpha\Psi' + \gamma\Psi + \frac{1}{2}\theta\Psi^2 = 0. \tag{9}$$

Balancing Ψ'' and Ψ^2 in Eq.(10) yields $M = 2$. Hence the solution can be expressed as

$$\Psi(\zeta) = a_0 + a_1 Q + a_2 Q^2, \quad Q = \frac{1}{1+\exp(\zeta)}. \tag{10}$$

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Substituting Eq.(10) in (9) and solving

$$a_0 = 0, \quad a_1 = \frac{24k_1\alpha}{5\theta}, \quad a_2 = -\frac{12k_1\alpha}{5\theta},$$
$$c_1 = \frac{\alpha}{5k_1}, \quad \gamma = \frac{\alpha - 6k_1^2\alpha}{5k_1}, \quad k_1 \neq 0.$$

and the exact soliton solutions of Eq.(1) as follows

$$u(x,t) = \Psi(\zeta) = \frac{24k_1\alpha}{5\theta}\left(\frac{1}{1+\exp\left(k_1\left(x - \frac{c_1 t^\beta}{\Gamma(\beta+1)}\right)\right)}\right) - \frac{12k_1\alpha}{5\theta}\left(\frac{1}{1+\exp\left(k_1\left(x - \frac{c_1 t^\beta}{\Gamma(\beta+1)}\right)\right)}\right)^2. \tag{11}$$

Implementation of Kansa-RBF collocation method

Consider the time-fractional OBBMB equation as

$$D_t^\beta u - D_t^\beta u_{xx} - \alpha u_{xx} + \gamma u_x + \theta u u_x = 0, \quad 0 < \beta \le 1, \; x \in \Omega \subset R, \; t > 0, \tag{12}$$

with the initial and boundary conditions

$$u(x,t) = g(x), \quad t = 0, \tag{13}$$

$$u(x,t) = h(x,t), \quad x \in \partial\Omega, \; t > 0, \tag{14}$$

In Eq.(12), the Caputo fractional derivative $\partial^\beta u(x,t)/\partial t^\beta$ can be expressed as

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} = \begin{cases} \dfrac{1}{\Gamma(1-\beta)} \displaystyle\int_0^t \frac{\partial u(x,\vartheta)}{\partial \vartheta} \frac{d\vartheta}{(t-\vartheta)^\beta}, & 0 < \beta < 1, \\ \dfrac{\partial u(x,t)}{\partial t}, & \beta = 1. \end{cases} \tag{15}$$

Let $u^n = u(x, t_n),\ t_n = n\Delta t,\ n = 0, 1, 2, ..., \mathcal{N}$. First, the time-fractional derivative in Eq.(12) is discretized by finite difference scheme [6] and substituting into (12), the time-discretized scheme between two successive time layers n and $n+1$ can be written in the following form:

$$\tau_0 u^{n+1} - \tau_0 \nabla_x^2 u^{n+1} - \alpha \nabla_x^2 u^{n+1} + \gamma \nabla_x u^{n+1}$$
$$= \begin{cases} \tau_0 u^n - \tau_0 \sum_{\ell=1}^{n} \delta_\ell (u^{n+1-\ell} - u^{n-\ell}) - \tau_0 \nabla_x^2 u^n \\ \quad + \tau_0 \sum_{\ell=1}^{n} \delta_\ell (\nabla_x^2 u^{n+1-\ell} - \nabla_x^2 u^{n-\ell}) - \theta u^n \nabla_x u^n, & n \geq 1, \\ \tau_0 u^0 - \tau_0 \nabla_x^2 u^0 - \theta u^0 \nabla_x u^0, & n = 0, \end{cases} \tag{16}$$

which implies

$$\tau_0 u^{n+1} - \tau_0 \nabla_x^2 u^{n+1} - \alpha \nabla_x^2 u^{n+1} + \gamma \nabla_x u^{n+1}$$
$$= \begin{cases} \tau_0 u^n - \tau_0 \chi_n - \tau_0 \nabla_x^2 u^n + \tau_0 \xi_n - \theta u^n \nabla_x u^n, & n \geq 1, \\ \tau_0 g - \tau_0 \nabla_x^2 g - \theta g \nabla_x g, & n = 0, \end{cases} \tag{17}$$

Continue...

where

$$\tau_0 = \frac{(\Delta t)^{-\beta}}{\Gamma(2-\beta)}, \quad \delta_\ell = (\ell+1)^{1-\beta} - \ell^{1-\beta}, \ \ell = 0, 1, 2, ..., n,$$

$$\chi_n = \sum_{\substack{\ell=1 \\ \ell \neq n}}^{n} \delta_\ell (u^{n+1-\ell} - u^{n-\ell}) + \delta_n (u^1 - g),$$

$$\xi_n = \sum_{\substack{\ell=1 \\ \ell \neq n}}^{n} \delta_\ell (\nabla_x^2 u^{n+1-\ell} - \nabla_x^2 u^{n-\ell}) + \delta_n (\nabla_x^2 u^1 - \nabla_x^2 g).$$

Now, using Kansa-RBF approach, $u^{n+1}(x)$ can be approximated as follows:

$$u^{n+1}(x) = \sum_{j=1}^{N} \eta_j^{n+1} \phi(r_j) + \eta_{N+1}^{n+1} x + \eta_{N+2}^{n+1}, \tag{18}$$

where $\{\eta_j^{n+1}\}$ are $(n+1)th$ time layer unknown coefficients.

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So, collocating Eq.(17) at N node points $x_i, i = 1, 2, ..., N$, then it follows

$$u^{n+1}(x_i) = \sum_{j=1}^{N} \eta_j^{n+1} \phi(r_{ij}) + \eta_{N+1}^{n+1} x_i + \eta_{N+2}^{n+1}, \qquad i = 1, 2, ..., N, \tag{19}$$

and the regularization conditions can be described as

$$\sum_{j=1}^{N} \eta_j^{n+1} = \sum_{j=1}^{N} \eta_j^{n+1} x_j = 0. \tag{20}$$

Plugging Eq.(19) into Eq.(17) and considering Eqs.(20) and (14), the discretized equation in matrix form can be illustrated as follows:

$$\mathbf{B}[\eta]^{n+1} = \rho^{n+1}, \tag{21}$$

where

Continue...

$$\mathbf{B} = \begin{bmatrix} J(\phi_{11}) & \cdots & J(\phi_{1j}) & \cdots & J(\phi_{1N}) & J(x_1) & J(1) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ J(\phi_{i1}) & \cdots & J(\phi_{ij}) & \cdots & J(\phi_{iN}) & J(x_i) & J(1) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ J(\phi_{N1}) & \cdots & J(\phi_{Nj}) & \cdots & J(\phi_{NN}) & J(x_N) & J(1) \\ x_1 & \cdots & x_j & \cdots & x_N & 0 & 0 \\ 1 & \cdots & 1 & \cdots & 1 & 0 & 0 \end{bmatrix}_{(N+2)\times(N+2)}, \tag{22}$$

where J represents an operator given by

$$J(*) = \begin{cases} (\tau_0 - \tau_0\nabla_x^2 - \alpha\nabla_x^2 + \gamma\nabla_x)(*), & 1 < i < N, \\ (*), & i = 1 \text{ or } i = N, \end{cases} \tag{23}$$

and $\rho^{n+1} = [\rho_1^{n+1} \quad \rho_2^{n+1} \quad \cdots \quad \rho_N^{n+1} \quad 0 \quad 0]^{\mathsf{T}}$,

$$\rho_i^{n+1} = \begin{cases} \tau_0 g - \tau_0\nabla_x^2 g - \theta g \nabla_x g\big|_{x=x_i}, & n = 0,\ 1 < i < N, \\ \tau_0 u^n - \tau_0\chi_n - \tau_0\nabla_x^2 u^n + \tau_0\xi_n - \theta u^n\nabla_x u^n\big|_{x=x_i}, & n \geq 1,\ 1 < i < N, \\ h(x_i, t_{n+1}), & i = 1 \text{ or } i = N. \end{cases} \tag{24}$$

Using Eq.(21), η_j^{n+1} can be computed and then from Eq.(19), the desired numerical solutions at each time layer can be obtained.

Numerical results

Example. Consider the time-fractional OBBMB equation as follows:

$$D_t^{\beta} u - D_t^{\beta} u_{xx} - \alpha u_{xx} + \gamma u_x + \theta u u_x = 0, \ \ -5 \leq x \leq 5, \ t > 0. \qquad (25)$$

The initial and boundary conditions can be extracted from the exact solution derived in Eq.(11). The parameters taken are $\gamma = (\alpha - 6k_1^2\alpha)/5k_1^2$, $\alpha = 0.5$, $\theta = 10$, $k_1 = 1$, $\beta = 0.95$, and $c = 0.1$.

$$L_2 = \| U_{exact} - U_{numerical} \|_2 = \sqrt{\Sigma_{i=1}^{N} (U_{exact}(\mathbf{x}_i, t) - U_{numerical}(\mathbf{x}_i, t))^2},$$

$$L_\infty = \| U_{exact} - U_{numerical} \|_\infty = \max_i |U_{exact}(\mathbf{x}_i, t) - U_{numerical}(\mathbf{x}_i, t)|.$$

Table 1: The L_2 and L_∞ errors with $\Delta x = 0.2,\ \Delta t = 0.02$.

t	$L_2(u)$	$L_\infty(u)$
0.1	8.43162E-3	1.98563E-3
0.2	8.63603E-3	2.02926E-3
0.3	8.85072E-3	2.08345E-3
0.4	9.07101E-3	2.14378E-3
0.5	9.29511E-3	2.21531E-3
0.7	9.75199E-3	2.36986E-3
0.7	9.75199E-3	2.36986E-3
0.8	9.98419E-3	2.45022E-3
0.9	1.02187E-2	2.53203E-3
1.0	1.04556E-2	2.61493E-3

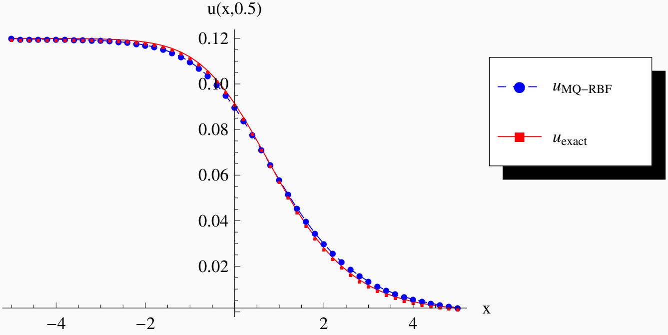

Fig. 1: Comparison of exact and MQ-RBF method solutions at $t = 0.5$.

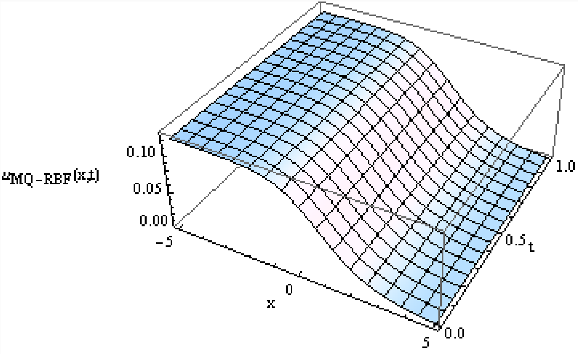

Fig. 2: The 3D surface solution plotted when $\Delta x = 0.2,\ \Delta t = 0.02$.

Conclusion

- The time-fractional OBBMB equation has been solved numerically using the Kansa-RBF collocation method, in which multiquadrics taken as RBF.
- To attain this, a numerical scheme based on finite difference and Kansa method has been proposed.
- The computational results show a high level of agreement with Kudryashov method solutions.
- The numerical investigations addressed in this work can be useful in the analysis of various nonlinear phenomena occurring in a wide range of scientific applications such as the long-wavelength surface waves in liquids, acoustic-gravity waves in compressible fluids, and hydromagnetic waves in cold plasma.
- Also, it can be inferred from the computational results that the proposed method is convenient to produce the fruitful numerical simulations for various types of nonlinear FPDEs.

- B Sagar and S. Saha Ray, 2021, "Numerical and analytical investigation for solutions of fractional Oskolkov-Benjamin-Bona-Mahony-Burgers equation describing propagation of long surface waves," International Journal of Modern Physics B, Vol. **35**, No. 32, 2150326 (World Scientific, SCI).

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Thank You