ISOCLINISM AND FACTOR SET IN REGULAR HOM-LIE SUPERALGEBRAS

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ABSTRACT

Hom-Lie superalgebras can be considered as the deformation of Lie super algebras; which are Z_2-graded generalization of Hom-Lie algebras. The motivation of this paper is to introduce the concept of isoclinism and factor set in regular Hom-Lie superalgebras. Moreover, we obtain that, two finite same dimensional regular Hom-Lie superalgebras are isoclinic if and only if they are isomorphic.

Keywords: Hom-Lie superalgebra, Isoclinism, Factor set.

Isoclinism and factor set in regular Hom-Lie Superalgebras

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Definition

A *superalgebra* is a vector superspace $L = L_{\bar{0}} \oplus L_{\bar{1}}$ endowed with an algebra structure such that $L_\alpha L_\beta \subseteq L_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{Z}_2$.

Definition

A *Lie superalgebra* is a vector superspace $L = L_{\bar{0}} \oplus L_{\bar{1}}$ with a bilinear mapping $[.,.] : L \times L \to L$ satisfying the following identities:

1. $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{Z}_2$ ($\mathbb{Z}_2$- grading),
2. $[x, y] = -(-1)^{|x||y|}[y, x]$ (graded skew-symmetry),
3. $(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0$ (graded Jacobi identity),

for $x, y, z \in L$.

Define the supercommutator bracket on L,

$$[x, y] = xy - (-1)^{|x||y|}yx \text{ for } x, y \in L.$$

We write the superdimensions of L as $sdim\ (L) = (m|n) = m + n$, where $sdim\ (L_{\bar{0}}) = m$ and $sdim\ (L_{\bar{1}}) = n$.

- **Kac** [1977] introduced the $\mathbb{Z}_2$-graded Lie algebras which are known as Lie superalgebras.
- **Nayak** [2018] studied isoclinism for Lie superalgebras.
- **Nayak et al.** [2019] gave the notion of factor set for Lie superalgebras.
- **Hartwig et al.** [2006] studied Hom-Lie algebras.
- **Padhan et al.** [2020] developed isoclinism and factor set for Hom-Lie algebras.
- **Ammar et al.** [2010] generalized Hom-Lie algebras to Hom-Lie superalgebras.

Definition

A *Hom-Lie superalgebra* is a triple $(L, [.,.], \alpha)$ which consists of a $\mathbb{Z}_2$-graded vector space L, an even bilinear map $[.,.] : L \times L \to L$ and an even homomorphism $\alpha : L \to L$ satisfying

1. $[m_1, m_2] = -(-1)^{|m_1||m_2|}[m_2, m_1]$ (graded skew-symmetry),
2. $(-1)^{|m_1||m_3|}[\alpha(m_1), [m_2, m_3]] + (-1)^{|m_3||m_2|}[\alpha(m_3), [m_1, m_2]] + (-1)^{|m_2||m_1|}[\alpha(m_2), [m_3, m_1]] = 0$ (graded Hom-Jacobi identity),

for homogeneous elements $m_1, m_2, m_3 \in L$.

• A graded subspace $W \subseteq L$ is a *Hom-subalgebra* of L if $\alpha(W) \subseteq W$ and W is closed under the bracket operation $[.,.]$, i.e. $[W, W] \subseteq W$. A Hom-subalgebra W is called a *ideal* of V if $[W, L] \subseteq W$. The ideal $L^{'} = [L, L]$ is called the *derived subalgebra*.

- The *center* of a Hom-Lie superalgebra $(L, [.,.], \alpha)$ is defined by

$$Z(L) = \{x \in L : [x, y] = 0 \ for \ y \in L\}.$$

- An *abelian* Hom-Lie superalgebra is a vector superspace L equipped with trivial bracket and a linear map $\alpha : L \to L$.

- A Hom-Lie superalgebra $(L, [.,.], \alpha)$ is called *multiplicative* if $\alpha[m_1, m_2] = [\alpha(m_1), \alpha(m_2)]$ for $m_1, m_2 \in L$. A multiplicative Hom-Lie superalgebra $(L, [.,.], \alpha)$ is said to be *regular* if α is bijective.

- A regular Hom-Lie superalgebra $(L, [.,.], \alpha)$ is called stem Hom-Lie superalgebra whenever $Z(L) \subseteq L'$.

Lemma

If $(L, [.,.], \alpha)$ is regular Hom-Lie superalgebra, then $Z(L)$ is an ideal of L.

• Let $(L, [.,.]_1, \alpha_1)$ and $(W, [.,.]_2, \alpha_2)$ be two Hom-Lie superalgebras. A *homomorphism* from $f : (L, [.,.]_1, \alpha_1) \rightarrow (W, [.,.]_2, \alpha_2)$ is a $\mathbb{F}$-linear map $f : L \rightarrow W$ such that $f([m_1, m_2]_1) = [f(m_1), f(m_2)]_2$ and $f\alpha_1 = \alpha_2 f$ for all $m_1, m_2 \in L$. They are isomorphic if $f : L \rightarrow W$ is bijective .

• For any ideal K of $(L, [.,.], \alpha)$, we can define quotient Hom-Lie superalgebra on the quotient vector superspace L/K by defining $[.,.] : L/K \times L/K \rightarrow L/K$ by

$$[\overline{m_1}, \overline{m_2}] = \overline{[m_1, m_2]} \textit{ for } \overline{m_1}, \overline{m_2} \in L/K,$$

and $\tilde{\alpha} : L/K \rightarrow L/K$ is induced by α, i.e. $\tilde{\alpha}(\overline{m}) = \alpha(m) + K$.

Definition

Let $(L, [.,.], \alpha_1)$ and $(W, [.,.], \alpha_2)$ be two regular Hom-Lie superalgebras, $\mu : \frac{L}{Z(L)} \to \frac{W}{Z(W)}$ and $\nu : L^{'} \to W^{'}$ be Hom-Lie superalgebra isomorphisms such that the following diagram is commutative:

$$
\begin{array}{ccc}
\frac{L}{Z(L)} \times \frac{L}{Z(L)} & \xrightarrow{\ \sigma\ } & L' \\
\Big\downarrow \mu^2 & & \Big\downarrow \nu \\
\frac{W}{Z(W)} \times \frac{W}{Z(W)} & \xrightarrow[\ \rho\ ]{} & W'
\end{array}
$$

where $\sigma(\overline{m_1}, \overline{m_2}) := [\overline{m_1}, \overline{m_2}]$ for $m_1, m_2 \in L$ and $\rho(\overline{n_1}, \overline{n_2}) := [\overline{n_1}, \overline{n_2}]$ where $n_1, n_2 \in W$. Then (μ, ν) is called *isoclinism*.

Definition

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a finite dimensional Hom-Lie superalgebra. The bilinear map;

$$r : L/Z(L) \times L/Z(L) \to Z(L),$$

is said to be a factor set if the following properties hold:

1. $r(\overline{m_1}, \overline{m_2}) \subseteq Z(L)_{\alpha+\beta}$, $\alpha, \beta \in \mathbb{Z}_2$,
2. $r(\overline{m_1}, \overline{m_2}) = -(-1)^{|\overline{m_1}||\overline{m_2}|} r(\overline{m_2}, \overline{m_1})$,
3. $r([\overline{m_1}, \overline{m_2}], \tilde{\alpha}(\overline{m_3})) = r(\tilde{\alpha}(\overline{m_1}), [\overline{m_2}, \overline{m_3}]) - (-1)^{|\overline{m_1}||\overline{m_2}|} r(\tilde{\alpha}(\overline{m_2}), [\overline{m_1}, \overline{m_3}])$,

for all homogeneous elements $\overline{m_1}, \overline{m_2}, \overline{m_3} \in L/Z(L)$, and $\tilde{\alpha}$ is an even homomorphism $\tilde{\alpha} : L/Z(L) \to L/Z(L)$ satisfying $\tilde{\alpha}(\overline{m}) = \alpha(m) + Z(L)$.

The factor set r is said to be multiplicative if

$$r(\tilde{\alpha}(\overline{m_1}), \tilde{\alpha}(\overline{m_2})) = \alpha r(\overline{m_1}, \overline{m_2}) \text{ for } \overline{m_1}, \overline{m_2} \in L/Z(L).$$

Lemma

A factor set r exists in such a way that $L \cong (Z(L), L/Z(L), r)$ for any regular Hom-Lie superalgebra $(L, [.,.], \alpha)$ of parity γ.

Lemma

Let $(L,[.,.],\alpha_1)$ be a stem Hom-Lie superalgebra in an isoclinism family of Hom-Lie superalgebras $\mathcal{C}$. Then for any stem Hom-Lie superalgebra $(W,[.,.],\alpha_2)$ of $\mathcal{C}$, there exists a factor set r over $(L,[.,.],\alpha_1)$ such that $W \cong (Z(L), L/Z(L), r)$.

Theorem

Let $(L,[.,.],\alpha_1)$ and $(W,[.,.],\alpha_2)$ be two finite dimensional stem Hom-Lie superalgebras having same parity. Then $L \sim W$ iff $L \cong W$.

Theorem

Let $\mathcal{C}$ be an isoclinism family of finite dimensional regular Hom-Lie superalgebras. Then any $L \in \mathcal{C}$ can be expressed as $L = T \oplus A$ where T is a stem Hom-Lie superalgebra and A is some finite dimensional abelian Hom-Lie superalgebra.

Theorem

If $(L,[.,.],\alpha_1)$ and $(W,[.,.],\alpha_2)$ be two regular Hom-Lie superalgebras with same dimension and having same parity. Then $L \sim W$ iff $L \cong W$.

Example

Let $(L,[.,.],\alpha_1)$ be a $(2|1)$ dimensional Hom-Lie superalgebra with the basis $\{e_1, e_2 \mid e_3\}$ and the commutator relations are defined by;

$$[e_1, e_3, e_3] = e_1,\ [e_2, e_3, e_3] = e_2,$$

and all other commutator relations are zero. Then $L^{'} =< e_1, e_2 >$ and $Z(L) = 0$ and hence, $L/Z(L) \cong L$.
Now, let $(W,[.,.],\alpha_2)$ be a $(3|1)$ dimensional Hom-Lie superalgebra with the basis $\{e_1', e_2', e_3' \mid e_4'\}$ and the commutator relations are defined by;

$$[e_1', e_4', e_4'] = e_1',\ [e_2', e_4', e_4'] = e_2',$$

and all other commutators are zero. Then $W' =< e_1', e_2' >$ and $Z(W) = \{e_3'\}$ and hence, $W/Z(W) = \{\overline{e_1'}, \overline{e_2'} \mid \overline{e_4'}\}$ where $\overline{e_i'} = e_i + Z(W)$ for $i = 1, 2, 4$.
A simple verification shows that $L^{'} \sim W^{'}$ and $\frac{L}{Z(L)} \cong \frac{W}{Z(W)}$ from which one can deduce that $L \sim W$ while $dim(L) \neq dim(W)$, i.e., L and W are not isomorphic.

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Thank You