# A note on fractal dimension for a class of fractal interpolation functions

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**Abstract**

The fractal interpolation functions with appropriate iterated function systems (IFSs) provide a method to perturb and approximate a continuous function on a compact interval $I$. This method produces a class of functions $f^\alpha \in \mathcal{C}(I)$, where $\alpha$ is a scale parameter. As essential parameters of the IFS, the scaling factors have important consequences in the properties of the function $f^\alpha$. Also, the interpolant or a certain derivative of it may have a non-integer box-counting dimension depending on the scaling factors magnitude. In this talk, we discuss an exact estimation of box dimension of $\alpha$-fractal functions under suitable hypotheses on IFSs.

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## Introduction to $\alpha$-fractal functions

- The concept of fractal interpolation function (FIF) using the theory of iterated function system (IFS) was first introduced by Barnsley [1].
- The most extensively studied FIFs so far are defined by the IFS: $L_i(x) = a_i x + b_i,\ F_i(x,y) = \alpha_i y + q_i(x)$.
- Let $f \in \mathcal{C}(I)$. For a fixed partition $\Delta := \{x_0, x_1, \ldots, x_N\}$ of $I = [x_0, x_N]$, Navascues [2] considered the maps $q_i(x) = f \circ L_i(x) - \alpha_i b(x)$, where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is the scaling vector and $b$ is the base function satisfying $b \neq f,\ b(x_0) = f(x_0), b(x_N) = f(x_N)$.
- The family of fractal function $\{f^\alpha : \alpha \in (-1,1)^{N-1}\}$, named as $\alpha$-fractal function interpolate and approximate $f$.
- The map $\mathcal{F}^\alpha : \mathcal{C}(I) \to \mathcal{C}(I)$ which sends $f$ to $f^\alpha$ is called $\alpha$-fractal operator. Furthermore, $f^\alpha_{\Delta,b}$ satisfies the self-referential equation

$$f^\alpha_{\Delta,b}(x) = f(x) + \alpha_j (L_j^{-1}(x)).(f^\alpha - b)\big(L_j^{-1}(x)\big) \quad \forall\ x \in [x_{j-1}, x_j],\ j \in \mathbb{N}_{N-1}. \tag{1}$$

[1] M. F. Barnsley, Fractal functions and interpolation, Constr. Approx., 2, (1986), 303-329

[2] M. A. Navascués, Fractal polynomial interpolation, Z. Anal. Anwendungen, 24(2), 2005, 401-418.

## Box-dimension

- Let $F$ be a nonempty bounded subset of $\mathbb{R}^n$ and let $N_\delta(F)$ denote the smallest number of sets of diameter less than or equal to $\delta$ which covers $F$.
- The lower and upper box-counting dimension of $F$ is defined as

$$\underline{\dim}_B(F) = \lim \inf_{\delta \to 0+} \frac{N_\delta(F)}{-\log \delta}, \ \overline{\dim}_B(F) = \lim \sup_{\delta \to 0+} \frac{N_\delta(F)}{-\log \delta}.$$

- The Hausdorff dimension of $F$ is denoted by $\dim_H(F)$ and for any bounded subset $F$ of $\mathbb{R}^n$,

$$\dim_H(F) \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F).$$

[3] [4]

[3] K. Falconer, Fractal Geometry, 2nd ed., John Wiley and Sons, Inc., Hoboken, NJ, Mathematical Foundations and Applications, 2003.

[4] P. R. Massopust, Fractal Functions, Fractal Surfaces, and Wavelets, Academic Press, Inc., San Diego, CA, 1994.

## Literature Review

There are different works on box dimension of fractal interpolations functions

- M. F. Barnsley, Fractal functions and interpolation, Constr. Approx., 2, (1986), 303-329.
- Deng, Guantie Hausdorff dimension of a fractal interpolation function. Colloq. Math., 99(2), (2004), 275-281.
- H. J. Ruan, W. Y. Su, and K. Yao, Box dimension and fractional integral of linear fractal interpolation functions, J. Approx. Theory 161(1), (2009), 187-197.
- Y. S. Liang, Box dimensions of Riemann-Liouville fractional integrals of continuous functions of bounded variation, Nonlinear Anal. 72 (2010), no. 11, 4304-4306.
- M. F. Barnsley and P. R. Massopust, Bilinear fractal interpolation and box dimension, J.Approx. Theory 192 (2015), 362-378.
- Md. N. Akhtar, M. G. P. Prasad, and M. A. Navascués, Box dimensions of $\alpha$-fractal functions, Fractals, 24(3), (2018) 1650037-13.
- S. Verma and P. Viswanathan, A revisit to $\alpha$-fractal function and box

## Results

- **Result:** Let $W_j = (L_j(x), F_j(x,y))$, where $L_j$ and $F_j$ are as described above. The map $W_j : I \times [-M, M] \to I \times [-M, M]$ is a contraction map with respect to the metric

$$d((x,y),(z,w)) = c_1|x-z| + c_2|(y-f^\alpha(x)) - (w-f^\alpha(z))| \; \forall (x,y),(z,w) \in I \times \mathbb{R},$$

where $c_1, c_2 > 0$ provided

$$\max\left\{a_j + \frac{2c_2 M k_{\alpha_j}}{c_1}, \|\alpha_j\|_\infty\right\} < 1$$

and $\alpha_j : I \to \mathbb{R}$ satisfies $|\alpha_j(x) - \alpha_j(y)| \le k_{\alpha_j}|x-y|$.

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[5] M. F. Barnsley and P. R. Massopust, Bilinear fractal interpolation and box dimension, J.Approx. Theory 192 (2015), 362-378.

**Theorem 1.**

*Let* $\mathcal{I} := \{I \times \mathbb{R}; W_1, W_2, \dots, W_{N-1}\}$ *be the IFS such that*

$$r_j \|(x,y) - (w,z)\|_2 \le \|W_j(x,y) - W_j(w,z)\|_2 \le R_j \|(x,y) - (w,z)\|_2,$$

*for every* $(x,y), (w,z) \in I \times \mathbb{R}$, *where* $0 < r_j \le R_j < 1 \ \forall\, j \in \{1, 2, \dots, N-1\}$. *Then* $s_* \le \dim_H(Graph(f^\alpha)) \le s^*$, *where* $s_*$ *and* $s^*$ *are determined by* $\sum_{j=1}^{N} r_j^{s_*} = 1$ *and* $\sum_{j=1}^{N} R_j^{s^*} = 1$ *respectively.*

**Remark 2.**

*In particular, with the notation in* [6] *we can omit the following condition from that theorem*

$$t_1.t_N \le (Min\{a_1, a_N\}) \Big( \sum_{n=1}^{N} t_n^l \Big)^{2/l}.$$

[6] M. F. Barnsley, Fractal functions and interpolation, Constr. Approx., 2, (1986), 303-329

## Oscillation space

- We define the total oscillation of order $m$,

$$Osc(m, f) = \sum_{|Q|=p^{-m}} R_f(Q),$$

  where the sum ranges over all $p$-adic intervals $Q \subset [0, 1]$ of length $|Q| = \frac{1}{p^m}$ and $R_f(Q) = \sup_{x,y \in Q} |f(x) - f(y)|$.
- Let $\beta \in \mathbb{R}$. The oscillation space $\mathcal{V}^\beta(I)$ is defined by

$$\mathcal{V}^\beta(I) = \left\{ f \in \mathcal{C}(I) : \sup_{m \in \mathbb{N}} \frac{Osc(m, f)}{p^{m(1-\beta)}} < \infty \right\}.$$

**Theorem 3.**

*Let $f, b, \alpha_j$ $(j \in J) \in \mathcal{V}^\beta(I)$ be such that $b(x_1) = f(x_1)$ and $b(x_N) = f(x_N)$. Further we assume that $|L_j(I)| = \frac{1}{p^{k_j}}$ for some $k_j \in \mathbb{N}$ with $\sum_{j \in J} \frac{1}{p^{k_j}} = 1$. For $\max\left\{ \|\alpha\|_\infty + \sum_{j \in J} \sup_{m \in \mathbb{N}} \frac{Osc(m, \alpha_j)}{p^{m(1-\beta)}}, \sum_{j \in J} \|\alpha_j\|_\infty \right\} < 1$, we have $f^\alpha \in \mathcal{V}^\beta(I)$.*

## Result

### Theorem 4.

[7] *Let $f$ be a real-valued continuous function defined on $I$, we have*

$$\overline{\dim}_B(Graph(f)) \leq 2-\gamma \iff f \in \mathcal{V}^{\gamma-}(I) \quad \text{if } 0<\gamma\leq 1$$

*and*

$$\overline{\dim}_B(Graph(f)) \geq 2-\gamma \iff f \in \mathcal{V}^{\gamma+}(I) \quad \text{if } 0\leq\gamma< 1.$$

### Remark 5.

*Let $0<\gamma\leq 1$ and $f, b, \alpha_j$ be suitable functions satisfying the hypothesis of Theorem 6. Then, Theorem 4 yields that $\overline{\dim}_B(Graph(f^\alpha)) \leq 2-\gamma$.*

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[7] A. Carvalho, Box dimension, oscillation and smoothness in function spaces, J. Funct. Spaces Appl., 3 (2005), 287-320.

## Hölder Space

- We define the Hölder space as

$$\mathcal{H}^s(I) := \{g : I \to \mathbb{R} : \text{ g is Hölder continuous with exponent } s\}.$$

- We use the norm $\|g\|_{\mathcal{H}} := \|g\|_\infty + [g]_s$, where

$$[g]_s = \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|^s}$$

.

**Theorem 6.**

*Let $f, b$ and $\alpha$ be Hölder continuous with exponent $s$ such that $b(x_1) = f(x_1)$ and $b(x_N) = f(x_N)$. Then with the notation $a := \min\{a_j : j \in J\}$ we have $f^\alpha$ is Hölder continuous with exponent $s$ provided $\frac{\|\alpha\|_{\mathcal{H}}}{a^s} < 1$.*

## Result

**Theorem 7.**

[8] *Let $f$ be a germ function, and $b, \alpha_j$ be suitable continuous functions such that*

$$\begin{aligned} |f(x) - f(y)| &\leq k_f |x-y|^s, \\ |b(x) - b(y)| &\leq k_b |x-y|^s, \\ |\alpha_j(x) - \alpha_j(y)| &\leq k_\alpha |x-y|^s \end{aligned} \tag{2}$$

*for every $x, y \in I, j \in J$, and for some $k_f, k_b, k_\alpha > 0, s \in (0,1]$. Further, assume that there are constants $K_f, \delta_0 > 0$ such that for each $x \in I$ and $\delta < \delta_0$ there exists $y \in I$ with $|x-y| \leq \delta,\ |f(x) - f(y)| \geq K_f |x-y|^s$ and $K_f - (\|b\|_\infty + M) a^{-s} k_\alpha > 0$. We have $\dim_B \big(Graph(f^\alpha)\big) = 2 - s$ provided that $\|\alpha\|_{\mathcal{H}} < a^s$ and $\|\alpha\|_\infty < \frac{K_f - (\|b\|_\infty + M) k_\alpha a^{-s}}{(k_{f,b,\alpha} + k_b) a^{-s}}$.*

[8] S. Jha, S. Verma, Dimensional analysis of $\alpha$-fractal function, Results Math. 186(4), (2021), 1-24.

### Remark 8.

*In* [9]*, Akhtar et al. computed the box dimension of* $\alpha$*-fractal function under certain condition. But for the Hölder exponent* $s \in (0,1)$ *the author has calculated an upper bound. In Theorem 7, we have obtained the exact estimation of the box dimension of* $\alpha$*-fractal function under suitable condition.*

### Theorem 9.

*Let* $f, \alpha_j \ (j \in J)$ *and* $b$ *be Hölder continuous with exponent* $s$ *such that* $b(x_1) = f(x_1)$ *and* $b(x_N) = f(x_N)$*. If* $\|\alpha\|_{\mathcal{H}} < a^s$ *with* $a = \min\{a_j : j \in J\}$ *then*

$$1 \leq \dim_H(Graph(f^{\alpha})) \leq 2 - s.$$

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[9]Md. N. Akhtar, M. G. P. Prasad, and M. A. Navascués, Box dimensions of $\alpha$-fractal functions, Fractals, 24(3), (2018) 1650037-13.

## Bounded Variation

- Let $\mathcal{BV}(I)$ denotes the set of all functions of bounded variation on $I$ and define a norm on $\mathcal{BV}(I)$ by $\|f\|_{\mathcal{BV}} := |f(t_0)| + V(f, I)$, where $V(f, I) = \sup_P \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$, the supremum is taken over all partitions $P$ of the interval $I$.
- **Result** In [10], Liang showed that if $f \in \mathcal{C}(I) \cap \mathcal{BV}(I)$, then $\dim_H(Graph(f)) = \dim_B(Graph(f)) = 1.$

**Theorem 10.**

*Let $f \in \mathcal{BV}(I)$. Suppose that $\triangle = \{x_1, x_2 \dots, x_N : x_1 < x_2 < \cdots < x_N\}$ is a partition of $I$, $b \in \mathcal{BV}(I)$ satisfying $b(x_1) = f(x_1),\ b(x_N) = f(x_N)$, and $\alpha_j\ (j \in J)$ are functions in $\mathcal{BV}(I)$ with $\|\alpha\|_{\mathcal{BV}} < \frac{1}{2(N-1)}$. Then $f^\alpha \in \mathcal{BV}(I)$.*

**Theorem 11.**

*Let $f, b$ be continuous functions of bounded variations and $\alpha_j$ $(j \in J)$ are functions of bounded variation with $\|\alpha\|_{\mathcal{BV}} < \frac{1}{2(N-1)}$. Then $\dim_H(Graph(f^\alpha)) = \dim_B(Graph(f^\alpha)) = 1.$*

[10] Y. S. Liang, Box dimensions of Riemann-Liouville fractional integrals of continuous functions of

## Absolute Continuous Space

Let $\mathcal{AC}(I)$ denotes the Banach space of all absolutely continuous functions on $I$ with its usual norm (denoted by $\|.\|_{\mathcal{AC}}$).

**Theorem 12.**

*Let $f \in \mathcal{AC}(I)$. Suppose that $\triangle = \{x_1, x_2, \dots, x_N : x_1 < x_2 < \cdots < x_N\}$ is a partition of $I$, $b \in \mathcal{AC}(I)$ satisfying $b(x_1) = f(x_1),\ b(x_N) = f(x_N)$ , and $\alpha_j\ (j \in J)$ are functions in $\mathcal{AC}(I)$ with $\|\alpha\|_{\mathcal{AC}} < \frac{a}{2(N-1)}$, where $a = \min\{a_j : j \in J\}$. Then, the fractal perturbation $f^\alpha$ corresponding to $f$ is absolutely continuous on $I$.*

**Theorem 13.**

*Let the germ function $f$ and the parameter $b$ be absolutely continuous functions. Suppose $\alpha_j$ $(j \in J)$ are absolutely continuous functions with $\|\alpha\|_{\mathcal{AC}} < \frac{a}{2(N-1)}$. Then $\dim_H(Graph(f^\alpha)) = \dim_B(Graph(f^\alpha)) = 1.$*

Thank you