Equidistant Paths in Graphs

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Abstract. Let \(n\) and \(k\) be integers, \(n \geq k \geq 1\). A graph \(G\) is said to admit property \(P_k\) if for any distinct pair \(x, y \in V(G)\), there exists \(k\) internally vertex disjoint paths between \(x\) and \(y\) of the same length. Consider the following family of graphs.

\[ G_n^k := \{ G_n : G_n \text{ admits property } P_k \}. \]

There are two interesting directions in the study of \(G_n^k\). Firstly, in the extremal direction, it is interesting to estimate the sparsity of graphs admitting property \(P_k\). That is, estimation of \(\nu(n, k) = \min \{|E(G_n)| : G_n \in G_n^k\}\). The other direction is structural: what properties in the graph ensures admittance of property \(P_k\). In this paper, we tackle the extremal question followed by some structural results on the same.

1 Introduction

Let \(n\) and \(k\) be integers, \(n \geq k \geq 1\). A graph \(G\) is said to admit property \(P_k\) if for any distinct pair \(x, y \in V(G)\), there exists \(k\) internally vertex disjoint paths between \(x\) and \(y\) of the same length. Consider the following family of graphs.

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2 Some initial observations

It is intimidate that the smallest graph that admits property \(P_2\) is the \(K_4\). In order to extend \(K_4\) to a larger graph admitting \(P_2\), one may try something like a mycielskian of \(K_4\): corresponding to \(\{v_1, v_2, v_3, v_4\}\) in the \(K_4\), add \(\{u_1, u_2, u_3, u_4\}\) to \(G\) such that \(V(G) = \{v_1, \ldots, v_4\} \cup \{u_1, \ldots, u_4\}\) and \(E(G) = E(K_4) \cup \{(u_i, v_j) : \{v_i, v_j\} \in E(K_4)\}\). It is not hard to verify that \(G\) admits property \(P_2\). Repeating the process, one may obtain a graph \(G_2^k\) on \(2^k\) vertices with \(6 \times 3^{k-1}\) edges. This observation yields the bound \(\nu(n, 2) \leq 2n^{1062}\). 3

One important observation from the above example is that in order to obtain sparser graphs admitting property \(P_2\), we need some structural symmetry in the arrangement of edges. But how much more can we improve \(\nu(n, 2)\)? As it turns out, the Hamming cube \(Q_{2^k}\) also admits \(P_2\). In fact, the Hamming cube \(Q_{2^k}\) admits \(P_{\lceil \frac{k}{2} \rceil}\). In order to see this, consider two points \((x_1, \ldots, x_k)\) and \((y_1, \ldots, y_k)\) at a Hamming distance \(t\) and without loss of generality,

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let the coordinates where the points differ are exactly 1, ..., t. We show t equidistant paths of length exactly t in the following manner. Let \( P_1 \) denote the path \((x_1, x_2, \ldots, x_t, \ldots, x_k) \rightarrow (y_1, x_2, \ldots, x_t, \ldots, x_k) \rightarrow \ldots \rightarrow (y_1, x_2, \ldots, x_t, \ldots, x_k)\). Let \( P_2 \) denote the path \((x_1, x_2, x_3, \ldots, x_t, \ldots, x_k) \rightarrow (x_1, y_2, x_3, \ldots, x_t, \ldots, x_k) \rightarrow \ldots \rightarrow (x_1, y_2, \ldots, y_t, \ldots, x_k)\). Similarly, let \( P_i \) denote the path where we start with \((x_1, x_2, \ldots, x_t, \ldots, x_k)\), then switch the \( i \)th coordinate followed by switching of the successive coordinates cyclically and ending at \((y_1, y_2, \ldots, x_{i-1}, y_i, \ldots, x_k) \rightarrow (y_1, y_2, \ldots, y_t, \ldots, x_k)\). It is easy to verify that each of the \( P_i \), \( 1 \leq i \leq t \) are vertex disjoint and of the same length \( t \).

Similarly, there are \( k - t \) equidistant paths of length exactly \( t + 2 \), where the first move and the last move is along a coordinate where \((x_1, \ldots, x_k)\) and \((y_1, \ldots, y_k)\) have the same value and internal points are all the correction in Hamming weights along that coordinate. This observation yields the bound \( \nu(n, \lceil k/2 \rceil) \leq n \log n \), where \( n = 2^k \).

A lower bound on \( \nu(n, k) \) can be obtained by the following simple observation that any graph on \( n \) vertices admitting property \( P_k \) must have connectivity at least \( k + 1 \); otherwise, two adjacent vertices in the graph can never have \( k \) equidistant vertex disjoint paths. This gives the following lower bound to \( \nu(n, k) \).

\[
\nu(n, k) \geq \frac{n(k + 1)}{2}.
\] (2)

The Hamming cube example gives an upper bound of \( 2nk \) for \( \nu(n, k) \) when \( n = 4^k \). We can improve the upper bound using grids in the following way. Consider the graph \( G_n \) where the vertices are the points on a \( n^{1/k} \times n^{1/k} \times \ldots \times n^{1/k} \) grid and a point \((x_1, x_2, \ldots, x_k)\) is adjacent to \( 2k \) points, namely \((x_1 \pm 1, x_2, \ldots, x_k), (x_1, x_2 \pm 1, \ldots, x_k), \ldots, (x_1, x_2, \ldots, x_k \pm 1)\).