APPEARANCE OF BALANCING AND RELATED NUMBER SEQUENCES IN STEADY STATE PROBABILITIES OF SOME MARKOV CHAINS

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MOTIVATION

Hlynka and Sajobi, in their paper ***"A Markov Chain Fibonacci Model "*** established the presence of Fibonacci numbers in numerators and denominators of the steady state probabilities of a particular class of Markov chains. Further they showed that a fixed Fibonacci type limiting vector can arise from more than one type of transition probability matrix. In addition to it the methods allow us to obtain limiting vectors for certain infinite state processes in a relatively easy manner, by working with properties of the finite state version. Motivated by their work, we construct a class of Markov chains such that their steady state distributions involve the balancing, Lucas-balancing and balancing-like numbers. An identity relating the balancing numbers and the silver ratio can be obtained as a byproduct.

ABSTRACT OF THE TALK

Balancing and Lucas-balancing numbers are solutions of a Diophantine equation and satisfy a second order homogeneous recurrence relation. Interestingly, these numbers can be seen as numerators and denominators in the steady state probabilities of a class of transition probability matrices of Markov chains. An identity relating the balancing numbers and the silver ratio can be obtained as a byproduct.

BALANCING NUMBERS

Balancing numbers $\boldsymbol{B}$ and balancers $\boldsymbol{R}$ are solutions of the Diophantine equation

$$\mathbf{1} + \mathbf{2} + \cdots + (\boldsymbol{B} - \mathbf{1}) = (\boldsymbol{B} + \mathbf{1}) + (\boldsymbol{B} + \mathbf{2}) + \cdots + (\boldsymbol{B} + \boldsymbol{R}).$$

Thus, $\mathbf{6}, \mathbf{35}$ and $\mathbf{204}$ are the first three balancing numbers with balancers $\mathbf{2}, \mathbf{14}$ and $\mathbf{84}$ respectively.

The definition of balancing numbers is due to Behera and Panda: *On the square roots of triangular numbers*, Fib. Quart., 37(1999), 98-205.

The concept of balancing numbers also coincides with the concept of numerical centers described in a paper by R. Finkelstein, *The house problem*, Amer. Math. Monthly, 72, 1965.

Notation: The $\boldsymbol{n^{th}}$ balancing number is denoted by $\boldsymbol{B_n}$.

The number

$$\boldsymbol{C_n = \sqrt{8B_n{}^2 + 1}}$$

is called the $\boldsymbol{n^{th}}$ Lucas balancing number.

Recurrence Relations

Balancing Numbers Satisfy

$$\boldsymbol{B_{n+1} = 6B_n - B_{n-1}}\ ;\ \boldsymbol{B_1 = 1, B_2 = 6.}$$

Lucas Balancing Numbers Satisfy

$$\boldsymbol{C_{n+1} = 6C_n - C_{n-1}}\ ;\ \boldsymbol{C_1 = 3, B_2 = 17.}$$

Balancing numbers behave like natural numbers:

$$B_1 + B_3 + \cdots + B_{2n-1} = {B_n}^2$$

and

$$B_2 + B_4 + \cdots + B_{2n} = B_n \cdot B_{n+1}$$

COBALANCING NUMBERS

Cobalancing numbers b and cobalancers r are solutions of the Diophantine equation

$$1+2+\cdots+b=(b+1)+(b+2)+\cdots+(b+r).$$

$2, 14$ and 84 are the first three cobalancing numbers with cobalancers $1, 6$ and 35 respectively.

The definition of cobalancing numbers is due to G.K. Panda and P.K. Ray, ***Cobalancing numbers and cobalancers*, Int. J. Math. Math. Sci. 8 (2005), 1189–1200.**

Cobalancing numbers

If $\boldsymbol{b}$ is a cobalancing number then $\mathbf{8}\boldsymbol{b}^2 + \mathbf{8}\boldsymbol{b} + \mathbf{1}$ is a perfect square and its positive square root is known as a Lucas-cobalancing numbers. The n-th cobalancing and Lucas-cobalancing numbers are denoted by b_n and c_n respectively and satisfy the binary recurrences

$$\boldsymbol{b}_{\boldsymbol{n}+\mathbf{1}} = \mathbf{6}\boldsymbol{b}_{\boldsymbol{n}} - \boldsymbol{b}_{\boldsymbol{n}-\mathbf{1}} + \mathbf{2},\ \boldsymbol{c}_{\boldsymbol{n}+\mathbf{1}} = \mathbf{6}\boldsymbol{c}_{\boldsymbol{n}} - \boldsymbol{c}_{\boldsymbol{n}-\mathbf{1}}$$

with initial terms $b_0 = b_1 = 0, c_0 = -1, c_1 = 1$.

Further, there is a sum formula relating balancing and cobalancing numbers: $\boldsymbol{B}_{\mathbf{1}} + \boldsymbol{B}_{\mathbf{2}} + \cdots + \boldsymbol{B}_{\boldsymbol{n}} = \frac{\boldsymbol{b}_{\boldsymbol{n}+\mathbf{1}}}{\mathbf{2}}$.

The Balancing-like sequences are recurrent sequences defined as

$$x_{n+1} = Ax_n - x_{n-1}\,; \;\; x_0 = 0, x_1 = 1 \; (A > 2)$$

The balancing-like sequence corresponding to $A = 6$ is the balancing sequence. Further, the balancing-like sequence with $A = 3$ coincides with the sequence of even indexed Fibonacci numbers. These sequences satisfies all important properties of the balancing numbers.

Markov chains

A discrete time Markov chain is a stochastic process $\{X_k\}$, where k runs over nonnegative integers, such that

$$Pr\{X_{k+1} = j | X_k = i, X_{k-1} = l, \dots, X_0 = r\} = Pr\{X_{k+1} = j | X_k = i\}.$$

In other words, the future state of the process depends on the present state and not on the past states.

Stationary transition probabilities

When the probability $Pr\{X_{k+1} = j | X_k = i\}$ depends only on i and j and not on k, then the Markov chain $\{X_k\}$ is said to have stationary transition probabilities.

Markov Chains

The matrix $\boldsymbol{P} = \left(\boldsymbol{P_{ij}}\right)$ is known as the (one step) transition probability matrix of the Markov chain $\{X_k\}$. The probability $P_{ij}^{(n)} = Pr\{X_{k+n} = j|X_k = i\}$ is known as an ***n*-step transition probability** which is the probability of passing from state i to state j in n transitions. The matrix $\boldsymbol{P^{(n)}} = \left(P_{ij}^{(n)}\right)$ is known as an n-step transition probability matrix. It is well-known that $\boldsymbol{P^{(n)}} = \boldsymbol{P^n}$.

Steady state probability vector

The limiting probability $\boldsymbol{\pi_j} = \lim_{\boldsymbol{n\to\infty}} \boldsymbol{P_{ij}^{(n)}}$ is known as a steady state probabilities of the Markov chain X_k.

Writting $\vec{\boldsymbol{\pi}} = (\boldsymbol{\pi_0}, \boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \dots)$**,** the steady state transition probabilities can be calculated from the relationships $\vec{\boldsymbol{\pi}} = \vec{\boldsymbol{\pi}}\boldsymbol{P}$ with $\boldsymbol{\pi_0} + \boldsymbol{\pi_1} + \boldsymbol{\pi_2} + \cdots = \boldsymbol{1}$.

BALANCING NUMBERS IN STEADY STATE PROBABILITIES

Consider a village that can accommodate a population of size not exceeding $n-1$. Let X_k be the population of the village corresponding to some discrete time k. The population of the village increases by one for each birth, decrease by one for each death and become zero when the whole population is migrated to a different destination.

Then $\{\boldsymbol{X_k}\}_{\boldsymbol{k=0}}^{\boldsymbol{n-1}}$ can be viewed as a Markov Chain and let us assign the transition probabilities as

$$\begin{cases} P_{i,i+1} = P_{i,i-1} = \dfrac{1}{6} \quad \text{if} \quad 1 \le i \le n-2, \\ P_{01} = \dfrac{1}{6}, P_{00} = P_{10} = P_{n-1,0} = \dfrac{5}{6}, \\ P_{i0} = \dfrac{2}{3}\ if\ 2 \le i \le n-2 \\ P_{ij} = 0\ ,\ \text{otherwise} \end{cases}$$

The transition probability matrix is given by

$$\boldsymbol{P} = \begin{bmatrix} 5/6 & 1/6 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 5/6 & 0 & 1/6 & 0 & 0 & \cdots & 0 & 0 \\ 2/3 & 1/6 & 0 & 1/6 & 0 & \cdots & 0 & 0 \\ 2/3 & 0 & 1/6 & 0 & 1/6 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 2/3 & 0 & 0 & 0 & 0 & \ddots & 0 & 1/6 \\ 5/6 & 0 & 0 & 0 & 0 & \ddots & 1/6 & 0 \end{bmatrix} \quad \textbf{....(1)}$$

and the vector $\vec{\boldsymbol{\pi}} = (\boldsymbol{\pi_0}, \boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \dots, \boldsymbol{\pi_{n-1}})$ representing the steady state probabilities of $\{\boldsymbol{X_k}\}_{\boldsymbol{k=0}}^{\infty}$ can be calculated from the relation

$$\vec{\pi} = \vec{\pi}\boldsymbol{P}$$

subject to the condition

$$\pi_0 + \pi_1 + \cdots + \pi_{n-1} = 1.$$

BALANCING NUMBERS IN STEADY STATE PROBABILITIES

Result I: *The steady state probability vector* $\vec{\boldsymbol{\pi}}$ *corresponding to the transition probability matrix* $\boldsymbol{P}$ *in* (1) *is given by*

$$\vec{\boldsymbol{\pi}} = \left(\frac{2B_n}{b_{n+1}}, \frac{2B_{n-1}}{b_{n+1}}, \cdots, \frac{2B_1}{b_{n+1}}\right)$$

where $\boldsymbol{B_n}$ *is the* n*-th balancing numbers and* $\boldsymbol{b_n}$ *is the* n-th *cobalancing numbers.*

The proof is as follows:

The equation $\vec{\pi} = \vec{\pi}\boldsymbol{P}$ gives

$$\pi_{n-1} = \frac{1}{6}\pi_{n-2}, \pi_i = \frac{1}{6}\pi_{i-1} + \frac{1}{6}\pi_{i+1}, i = 1,2, \dots, n-2.$$

and on rearranging, we get

$$\pi_{n-2} = 6\pi_{n-1}, \pi_{i-1} = 6\pi_i - \pi_{i+1}, i = 1,2, \dots, n-2.$$

We ignoring the first equation

$$\pi_0 = \frac{5}{6}\pi_0 + \frac{5}{6}\pi_1 + \frac{2}{3}(\pi_2 + \cdots + \pi_{n-2}) + \frac{5}{6}\pi_{n-1}$$

and in stead use

$$\pi_0 + \pi_1 + \cdots + \pi_{n-1} = 1$$

THE PROOF CONTINUES…

Now setting $\pi_{n-1} = k$, one can rewrite the system of equations as

$$\begin{cases} \pi_{n-1} = k = kB_1 \\ \pi_{n-2} = 6k = kB_2 \\ \pi_{i-1} = 6\pi_i - \pi_{i+1}, i = 1,2,\dots,n-2 \end{cases} \qquad \text{.......(2)}$$

The main aim is to show that $\pi_i = kB_{n-i}$ for $i = 0,1,2,\dots,n-1$ using mathematical induction.

THE PROOF CONTINUES…

From (2) we can see that the assertion is true for $i = 0,1$. Assume that the assertion is true for $i = j \leq n-1$, i.e., $\pi_j = kB_{n-j}$. In view of the equation

$$\boldsymbol{\pi_{j-1} = 6\pi_j - \pi_{j+1}},$$

we have

$$\boldsymbol{\pi_{j+1} = 6kB_{n-j} - kB_{n-j+1} = k\left(6B_{n-j} - B_{n-j+1}\right)}$$

and by the recurrence relation of balancing numbers, it follows that

$$\boldsymbol{\pi_{j+1} = kB_{n-j-1} = kB_{n-(j+1)}}$$

and the assertion is true for $i = j + 1 \leq n - 1$.

THE PROOF CONTINUES…

Further, $\sum_{i=0}^{n-1} \pi_i = 1$ implies that

$$k = \frac{1}{\sum_{l=1}^{n} B_l}.$$

Hence,

$$\pi_i = \frac{2B_{n-i}}{\sum_{l=1}^{n} B_l} \; for \; i = 0,1,\cdots,n-1.$$

Since $\sum_{l=1}^{n} B_l = \frac{b_{n+1}}{2}$, the proof is complete. ■

Another Transition Probability Matrix Leading to the Same Steady State Probabilities

Result II: If $0 < q \leq 1/6$, *then transition probability matrix*

$$\boldsymbol{P}(q) = \begin{bmatrix} 1-q & q & 0 & 0 & 0 & \cdots & 0 & 0 \\ 5q & 1-6q & q & 0 & 0 & \cdots & 0 & 0 \\ 4q & q & 1-6q & q & 0 & \cdots & 0 & 0 \\ 4q & 0 & q & 1-6q & q & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 4q & 0 & 0 & 0 & 0 & \ddots & 1-6q & q \\ 5q & 0 & 0 & 0 & 0 & \ddots & q & 1-6q \end{bmatrix}$$

results in the same steady state probabilities given by

$$\vec{\boldsymbol{\pi}} = \left(\frac{\boldsymbol{2B_n}}{\boldsymbol{b_{n+1}}}, \frac{\boldsymbol{2B_{n-1}}}{\boldsymbol{b_{n+1}}}, \cdots, \frac{\boldsymbol{2B_1}}{\boldsymbol{b_{n+1}}}\right).$$

Another Transition Probability Matrix with Balancing Numbers in the Steady State Probabilities

Result III: Similarly, the steady state probability vector corresponding to the following transition probability matrix

$$\boldsymbol{P} = \begin{bmatrix} 5/6 & 1/6 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 5/6 & 0 & 1/6 & 0 & 0 & \cdots & 0 & 0 \\ 2/3 & 1/6 & 0 & 1/6 & 0 & \cdots & 0 & 0 \\ 2/3 & 0 & 1/6 & 0 & 1/6 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 2/3 & 0 & 0 & 0 & 0 & \ddots & 0 & 1/6 \\ 2/3 & 0 & 0 & 0 & 0 & \ddots & 1/6 & 1/6 \end{bmatrix},$$

is given by

$$\vec{\pi} = (\pi_0, \pi_1, \pi_2, \dots, \pi_{n-1})$$

where
$$\pi_i = \frac{B_{n+i} - B_{n+i-1}}{B_n} \; for \; i = 0,1,\cdots,n-1.$$

LUCAS-BALANCING NUMBERS IN STEADY STATE PROBABILITIES

Result IV: The steady state probability vector corresponding to the transition probability matrix

$$\boldsymbol{P} = \begin{bmatrix} 5/6 & 1/6 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 5/6 & 0 & 1/6 & 0 & 0 & \cdots & 0 & 0 \\ 2/3 & 1/6 & 0 & 1/6 & 0 & \cdots & 0 & 0 \\ 2/3 & 0 & 1/6 & 0 & 1/6 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 2/3 & 0 & 0 & 0 & 0 & \ddots & 0 & 1/6 \\ 1/3 & 0 & 0 & 0 & 0 & \ddots & 1/6 & 1/2 \end{bmatrix}$$

is given by $\vec{\pi} = (\pi_0, \pi_1, \pi_2, \dots, \pi_{n-1})$ where $\pi_i = \frac{C_{n-i}}{\sum_{i=1}^{n} C_i}$ for $i = 0,1,\cdots, n-1$ and C_i denotes the i-th Lucas-balancing number.

Another Transition Probability Matrix with Lucas Balancing Numbers in the Steady State Probabilities

Result V: If q is a real number such that $0 < q \leq 1/6$, then the transition probability matrix

$$\boldsymbol{P}(q) = \begin{bmatrix} 1-q & q & 0 & 0 & 0 & \cdots & 0 & 0 \\ 5q & 1-6q & q & 0 & 0 & \cdots & 0 & 0 \\ 4q & q & 1-6q & q & 0 & \cdots & 0 & 0 \\ 4q & 0 & q & 1-6q & q & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 4q & 0 & 0 & 0 & 0 & \ddots & 1-6q & q \\ 2q & 0 & 0 & 0 & 0 & \ddots & q & 1-3q \end{bmatrix}$$

results in the same steady state probabilities

$$\pi_i = \frac{C_{n-i}}{\sum_{i=1}^{n} C_i} \text{ for } i = 0,1,\cdots,n-1.$$

LUCAS-COBALANCING NUMBERS IN STEADY STATE PROBABILITIES

Result VI: The steady state probability vector corresponding to the transition probability matrix $\boldsymbol{P} = (\boldsymbol{P_{ij}})$,

$$\boldsymbol{P} = \begin{bmatrix} 5/6 & 1/6 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 5/6 & 0 & 1/6 & 0 & 0 & \cdots & 0 & 0 \\ 2/3 & 1/6 & 0 & 1/6 & 0 & \cdots & 0 & 0 \\ 2/3 & 0 & 1/6 & 0 & 1/6 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 2/3 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 16/21 & 0 & 0 & 0 & 0 & \ddots & 0 & 1/14 \\ 1/3 & 0 & 0 & 0 & 0 & \ddots & 1/6 & 1/2 \end{bmatrix}$$

is given by

$$\vec{\pi} = \left(\frac{c_n}{\sum_{i=1}^{n} c_i}, \frac{c_{n-1}}{\sum_{i=1}^{n} c_i}, \cdots, \frac{c_1}{\sum_{i=1}^{n} c_i} \right)$$

where c_n *denotes the* n^{th} *Lucas-cobalancing number.*

SILVER RATIO IN STEADY STATE PROBABILITIES OF MARKOV CHAINS WITH INFINITE STATE SPACES

***Result VII*:** In this section, we consider a Markov chain having the infinite state space $\{0,1,2,\ldots\}$ and transition probability matrix $P = \left(P_{ij}\right)$

$$\boldsymbol{P} = \begin{bmatrix} 5/6 & 1/6 & 0 & 0 & 0 & \cdots \\ 5/6 & 0 & 1/6 & 0 & 0 & \cdots \\ 2/3 & 1/6 & 0 & 1/6 & 0 & \cdots \\ 2/3 & 0 & 1/6 & 0 & 1/6 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \ldots\ldots \textbf{(3)}$$

The steady state probability vector corresponding is $\vec{\pi} = (\pi_0, \pi_1, \pi_2, \ldots)$ where

$$\pi_i = \beta^i - \beta^{i+1}, i = 0,1,2, \ldots \text{ and } \beta = 3 - 2\sqrt{2}.$$

[Observe that $\beta = \frac{1}{\left(1+\sqrt{2}\right)^2}$ and the ratio $1 + \sqrt{2}: 1$ is known as the silver ratio.]

The proof proceeds as follows:

Using the identity $\vec{\boldsymbol{\pi}} = \vec{\boldsymbol{\pi}}\boldsymbol{P}$, the balance equations for the calculation of steady state probabilities are given by

$$\begin{cases} \pi_0 = \frac{5}{6}\pi_0 + \frac{5}{6}\pi_1 + \frac{2}{3}(\pi_2 + \pi_3 + \cdots), \\ \pi_i = \frac{1}{6}\pi_{i-1} + \frac{1}{6}\pi_{i+1} \text{ for } i \geq 1. \end{cases}$$

On simplification, we get

$$\begin{cases} \pi_1 = 5\pi_0 - 4, \\ \pi_2 = 29\pi_0 - 24, \\ \pi_3 = 169\pi_0 - 140 \end{cases}$$

and using mathematical induction, it can be seen that

$$\pi_i = (B_{i+1} - B_i)\pi_0 - 4B_i, i = 1,2, \ldots \qquad \ldots \quad (4)$$

PROOF CONTINUES…

Further from Result III, it can be seen that

- $\pi_0 = \lim_{n\to\infty} \frac{B_n - B_{n-1}}{B_n} = 1 - \lim_{n\to\infty} \frac{B_{n-1}}{B_n} = 1 - \left(3 - 2\sqrt{2}\right) = 1 - \beta$
- $\pi_1 = \lim_{n\to\infty} \frac{B_{n-1} - B_{n-2}}{B_n} = \lim_{n\to\infty} \left(\frac{B_{n-1}}{B_n} - \frac{B_{n-2}}{B_{n-1}} \cdot \frac{B_{n-1}}{B_n}\right) = \beta - \beta^2,$
- $\pi_2 = \lim_{n\to\infty} \frac{B_{n-2} - B_{n-3}}{B_n} = \beta^2 - \beta^3,$

and in general

$$\boldsymbol{\pi_i = \lim_{n\to\infty} \frac{B_{n-i-1} - B_{n-i}}{B_n} = \beta^i - \beta^{i+1}, i = 0, 1, 2, \dots}$$

A balancing identity using steady state probabilities

$\boldsymbol{\beta^{n+1} = \beta B_{n+1} - B_n, n = 1, 2, \ldots}$ *where* $\boldsymbol{\beta = 3 - 2\sqrt{2}}$.

From the previous calculation, $\pi_0 = 1 - \beta$ and $\pi_i = \beta^i - \beta^{i+1} = \beta^i(1-\beta)$. Also from (4) we have

$$\pi_i = (B_{i+1} - B_i)\pi_0 - 4B_i, i = 1,2, \ldots$$

Substituting the value of π_0, we get

$$\beta^i(1-\beta) = (B_{i+1} - B_i)(1-\beta) - 4B_i.$$

Thus,

$$\beta^i = (B_{i+1} - B_i) - \frac{4B_i}{1-\beta} = B_{i+1} - \left(3 + 2\sqrt{2}\right)B_i = B_{i+1} - \frac{B_i}{\beta}$$

from which the above identity follows.

BALANCING-LIKE NUMBERS IN THE STEADY STATE PROBABILITIES OF MARKOV CHAINS

Result VIII: For the $n \times n$ *transition probability matrix* $\boldsymbol{P}(A)$

$$\boldsymbol{P}(A) = \begin{bmatrix} 1-\frac{1}{A} & \frac{1}{A} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1-\frac{1}{A} & 0 & \frac{1}{A} & 0 & 0 & \cdots & 0 & 0 \\ 1-\frac{2}{A} & \frac{1}{A} & 0 & \frac{1}{A} & 0 & \cdots & 0 & 0 \\ 1-\frac{2}{A} & 0 & \frac{1}{A} & 0 & \frac{1}{A} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 1-\frac{2}{A} & 0 & 0 & 0 & 0 & \ddots & 0 & \frac{1}{A} \\ 1-\frac{1}{A} & 0 & 0 & 0 & 0 & \ddots & \frac{1}{A} & 0 \end{bmatrix}$$

the steady state probability vector is given by $\vec{\pi} = (\pi_0, \pi_1, \pi_2, \ldots, \pi_{n-1})$ *where*

$$\pi_i = \frac{x_{n-i}}{\sum_{i=1}^{n} x_i} \; for \; i = 0,1,\cdots,n-1.$$

Another Transition Probability Matrix with Balancing-Like Numbers in the Steady State Probabilities

Result IX: If q is any real number such that $0 < q \leq \frac{1}{A}$, then the transition probability matrix

$$\boldsymbol{P}(q) = \begin{bmatrix} 1-q & q & 0 & 0 & 0 & \cdots & 0 & 0 \\ (A-1)q & 1-Aq & q & 0 & 0 & \cdots & 0 & 0 \\ (A-2)q & q & 1-Aq & q & 0 & \cdots & 0 & 0 \\ (A-2)q & 0 & q & 1-Aq & q & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ (A-2)q & 0 & 0 & 0 & 0 & \ddots & 1-Aq & q \\ (A-1)q & 0 & 0 & 0 & 0 & \ddots & q & 1-Aq \end{bmatrix},$$

gives the same steady state probabilities

$$\boldsymbol{\pi_i = \frac{x_{n-i}}{\sum_{i=1}^{n} x_i} \; for \; i = 0, 1, \cdots, n-1.}$$

CONCLUSION

In this work, we established the appearance of balancing and related numbers sequence in the steady state probabilities of some Markov chains. We also noticed that, in many instances, a class of transition probability matrices gives rise to same steady state probabilities. Using the balance equations, we also derived an identity relating the balancing numbers and the silver ratio. Some problems in this area are still open. We encourage the readers to construct transition probability matrices whose steady state vectors would explore some other number sequences. In this process, they may be able to prove some identities using the balance equations.

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ANY QUESTIONS?