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SOME FASCINATING PROPERTIES OF BALANCING NUMBERS

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Abstract: The study of number sequences has been a source of attraction to the mathematicians since ancient times. Since then many of them are focusing their interest on the study of the fascinating triangular numbers. In a recent study Behera and Panda tried to find the solutions of the Diophantine equation \(1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)\) and found that the square of any \(n \in \mathbb{Z}^+\) satisfying this equation is a triangular number. It can also be shown that if \(r \in \mathbb{Z}^+\) satisfies the above equation then \(r^2 + r\) is also a triangular number. If a pair \((n, r)\) constitutes a solution of the above equation then \(n\) is called a balancing number and \(r\) is called the balancer corresponding to the balancing number \(n\). In the joint paper “On the square roots of triangular numbers” published in “The Fibonacci Quarterly” in 1999, Behera and Panda introduced balancing numbers and studied many important properties of these numbers. In this paper we establish some other interesting arithmetic-type, de-Moivre’s-type and trigonometric-type properties of balancing numbers. We also establish a most important property concerning the greatest common divisor of two balancing numbers.

1. INTRODUCTION

Recently, Behera and Panda [3] introduced balancing numbers \(n \in \mathbb{Z}^+\) as solutions of the equation
\[
1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),
\]
calling \(r \in \mathbb{Z}^+\), the balancer corresponding to the balancing number \(n\). For example 6, 35 and 204 are balancing numbers with balancers 2, 14 and 84 respectively. It is also proved in [3] that a positive integer \(n\) is a balancing number if and only if \(n^2\) is a triangular number, that is \(8n^2 + 1\) is a perfect square. Though the definition of balancing number suggests that it must be greater than 2, Behera and Panda [3] accepted 1 as a balancing number being the positive square root of the square triangular number 1.

Behera and Panda [3], while accepting 1 as a balancing number, have set \(B_0 = 1, B_1 = 6\), and so on, using the symbol \(B_n\) for the \(n^{th}\) balancing number. To standardize the notation at par with Fibonacci numbers, we relabel the balancing numbers by setting \(B_1 = 1, B_2 = 6\) and so on.
Some results established by Behera and Panda [3] can be stated with this new convention as follows:

The second order linear recurrence:
\[ B_{n+1} = 6B_n - B_{n-1}; \quad n = 2, 3, \ldots \] ...(1)

The non-linear first order recurrence:
\[ B_{n+1} = 3B_n + \sqrt{8B_n^2 + 1}; \quad n = 1, 2, \ldots \] ...(2)

The relation:
\[ B_n = B_{r+1} \cdot B_{n-r} - B_r \cdot B_{n-r-1}; \quad r = 1, 2, \ldots, n-2. \] ...(3)

The Binet form:
\[ B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \quad n = 1, 2, \ldots \] ...(4)

where \( \lambda_1 = 3 + \sqrt{8} \) and \( \lambda_2 = 3 - \sqrt{8} \).

The interesting relation:
\[ B_{n+1} \cdot B_{n-1} = (B_n + 1)(B_n - 1). \] ...(5)

Now using (1) we can set \( B_0 = B_2 - 6B_1 = 6 - 6 \times 1 = 0. \)

In the next section we establish some arithmetic-type properties and other interesting properties of balancing numbers.

2. SOME INTERESTING RESULTS ON BALANCING NUMBERS

Throughout this section \( F_n \) is the \( n \)th Fibonacci number, \( L_n \) is the \( n \)th Lucas number, \( B_n \) is the \( n \)th Balancing number and \( C_n = \sqrt{8B_n^2 + 1} \) where \( n \in \mathbb{Z}^+ \). Some of the following results suggest that \( C_n \) is associated with \( B_n \) in the way \( L_n \) is associated with \( F_n \).

We know that if \( x \) and \( y \) are real or complex numbers, then \( (x + y)(x - y) = x^2 - y^2 \). In the following theorem we prove an analogous property of balancing numbers. This theorem also generalizes equation (5).

**Theorem 2.1:** If \( m \) and \( n \) are natural numbers and \( m > n \), then
\[ (B_m + B_n)(B_m - B_n) = B_{m+n} \cdot B_{m-n}. \]

**Proof:** Using the Binet form (4) and keeping in mind that \( \lambda_1 \lambda_2 = 1 \), we have
\[
B_{m+n} \cdot B_{m-n} = \frac{(\lambda_1^{m+n} - \lambda_2^{m+n})(\lambda_1^{m-n} - \lambda_2^{m-n})}{(\lambda_1 - \lambda_2)^2}
\]
\[
= \frac{(\lambda_1^{2m} + \lambda_2^{2m}) - (\lambda_1^{m+n} \lambda_2^{m-n} + \lambda_1^{m-n} \lambda_2^{m+n})}{(\lambda_1 - \lambda_2)^2}
\]
\[
= \frac{(\lambda_1^{2m} + \lambda_2^{2m}) - (\lambda_1^{2n} + \lambda_2^{2n})}{(\lambda_1 - \lambda_2)^2}
\]
\[
= \frac{(\lambda_1^{2m} + \lambda_2^{2m} - 2\lambda_1^m \lambda_2^m) - (\lambda_1^{2n} + \lambda_2^{2n} - 2\lambda_1^n \lambda_2^n)}{(\lambda_1 - \lambda_2)^2}
\]
\[
= \left[ \frac{\lambda_1^m - \lambda_2^m}{\lambda_1 - \lambda_2} \right]^2 - \left[ \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \right]^2
\]
\[
= B_m^2 - B_n^2 = (B_m + B_n)(B_m - B_n).
\]

**Remark:** The Fibonacci numbers satisfy a similar property (see [4], p.59)

\[
F_{m+n} \cdot F_{m-n} = F_m^2 - (-1)^{m+n} F_n^2.
\]

The identity of Theorem 1 looks more symmetric than this result.

We know that if \( n \) is a natural number, then \( 1 + 3 + \cdots + (2n - 1) = n^2 \), \( 2 + 4 + \cdots + 2n = n(n + 1) \) and \( 1 + 2 + \cdots + 2n = n(2n + 1) \). In the following theorem we prove three properties of balancing numbers similar to the above three identities.

**Theorem 2.2:** If \( n \) is a natural number then

(a) \( B_1 + B_3 + \cdots + B_{2n-1} = B_n^2 \),

(b) \( B_2 + B_4 + \cdots + B_{2n} = B_n B_{n+1} \),

(c) \( B_1 + B_2 + \cdots + B_{2n} = B_n (B_n + B_{n+1}) \).

**Proof:** From Theorem 2.1 we have

\[
B_{m+n} \cdot B_{m-n} = B_m^2 - B_n^2
\]

where \( m > n \). Replacing \( m \) by \( n + 1 \) in the above identity and keeping in mind that \( B_1 = 1 \) we obtain

\[
B_{2n+1} = B_{n+1}^2 - B_n^2,
\]

from which (a) follows.

Replacing \( n \) by \( 2n \) and \( r \) by \( n \) in equation (3) we find

\[
B_{2n} = B_{n+1} \cdot B_n - B_n B_{n-1},
\]

from which (b) follows.

The identity (c) directly follows from (a) and (b).
The complex identity \((\cos x + i \sin x)^n = \cos nx + i \sin nx\) is known as the de-Moivre’s formula (see [1]). The following theorem looks like de-Moivre’s formula.

**Theorem 2.3:** If \(n\) and \(r\) are natural numbers, then \((C_n + \sqrt{8}B_n)^r = C_{nr} + \sqrt{8}B_{nr}\).

**Proof:** Using the Binet form (4) we obtain

\[
C_n^2 = 8B_n^2 + 1 = 8 \left( \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \right)^2 + 1 = 8 \left( \frac{\lambda_1^{2n} + \lambda_2^{2n} - 2}{(2\sqrt{8})^2} \right) + 1 = \frac{\lambda_1^{2n} + \lambda_2^{2n} + 2}{4} = \left( \frac{\lambda_1^n + \lambda_2^n}{2} \right)^2.
\]

Hence

\[
C_n = \frac{\lambda_1^n + \lambda_2^n}{2}.
\]

Now

\[
C_n + \sqrt{8}B_n = \frac{\lambda_1^n + \lambda_2^n}{2} + \sqrt{8} \frac{\lambda_1^n - \lambda_2^n}{2\sqrt{8}} = \lambda_1^n.
\]

Thus

\[
(C_n + \sqrt{8}B_n)^r = (\lambda_1^n)^r = \lambda_1^{nr} = C_{nr} + \sqrt{8}B_{nr}.
\]

**Remark:** The Fibonacci numbers satisfy a similar property

\[
\left[ \frac{L_n + \sqrt{5}F_n}{2} \right]^r = \frac{L_m + \sqrt{5}F_m}{2}.
\]

**Corollary 2.4:** If \(n\) and \(r\) are natural numbers, then \((C_n - \sqrt{8}B_n)^r = C_{nr} - \sqrt{8}B_{nr}\).

**Proof:** Since

\[
C_n - \sqrt{8}B_n = \frac{\lambda_1^n + \lambda_2^n}{2} - \sqrt{8} \frac{\lambda_1^n - \lambda_2^n}{2\sqrt{8}} = \lambda_2^n,
\]

the result follows.

The following theorem looks like the trigonometric identity

\[
\sin(x + y) = \sin x \cos y + \cos x \sin y.
\]

**Theorem 2.5:** If \(m\) and \(n\) are natural numbers, then \(B_{m+n} = B_mB_n + C_mB_n\).
Proof: If $m$ and $n$ are natural numbers, then using equation (6) we obtain

$$(C_m + \sqrt{8} B_m)(C_n + \sqrt{8} B_n) = \lambda_1^m \lambda_1^n = \lambda_1^{m+n}$$

$$= (C_{m+n} + \sqrt{8} B_{m+n}).$$

...(7)

On the other hand,

$$(C_m + \sqrt{8} B_m)(C_n + \sqrt{8} B_n)$$

$$= (C_m C_n + 8 B_m B_n) + \sqrt{8}(B_m C_n + 8 C_m B_n).$$

...(8)

Comparing equations (7) and (8) we get

$$C_{m+n} + \sqrt{8} B_{m+n} = (C_m C_n + 8 B_m B_n) + \sqrt{8}(B_m C_n + 8 C_m B_n).$$

...(9)

Equating the rational and irrational parts from both sides of equation (9) we obtain

$$C_{m+n} = C_m C_n + 8 B_m B_n$$

and

$$B_{m+n} = B_m C_n + C_m B_n.$$

Remark: The corresponding property for Fibonacci numbers

$$F_{m+n} = \frac{1}{2} [F_m L_n + L_m F_n],$$

does not look like the trigonometric identity $\sin(x + y) = \sin x \cos y + \cos x \sin y.$

The following corollary looks like the trigonometric identity $\sin(x - y) = \sin x \cos y - \cos x \sin y.$

**Corollary 2.6:** If $m$ and $n$ are natural numbers and $m > n$, then $B_{m-n} = B_m C_n - C_m B_n.$

Proof: Same as Theorem 2.5.

The following corollary resembles the trigonometric identity $\sin 2x = 2 \sin x \cos x.$

**Corollary 2.7:** If $n$ is a natural number, then $B_{2n} = 2 B_n C_n.$

Proof: Directly follows from Theorem 2.5 with $m = n.$

Remark: The corresponding property for Fibonacci numbers $F_{2n} = F_n L_n$ (see [4]) does not look like $\sin 2x = 2 \sin x \cos x.$

For any two integers $m$ and $n$, let us denote the greatest common divisor of $m$ and $n$ by $(m, n).$ We know that $F_m$ divides $F_n$ if and only if $m$ divides $n$ and $(F_m, F_n) = F_{(m, n)}.$ The following results show that the balancing numbers also enjoy these beautiful properties.

**Theorem 2.8:** If $m$ and $n$ are natural numbers, then $B_m$ divides $B_n$ if and only if $m$ divides $n.$
To prove Theorem 2.8 we need the following lemmas.

**Lemma 2.9:** If m and n are natural numbers, then \((B_n^n, C_n^n) = 1\).

**Proof:** Since \(C_n^n = 8B_n^n + 1\), it follows that \((B_n^n, C_n^n) = 1\) and thus \((B_n^n, C_n^n) = 1\).

**Lemma 2.10:** If n and k are natural numbers, then \(B_k^k\) divides \(B_{nk}^k\).

**Proof:** The proof is based on induction. The hypothesis is trivial for \(n = 1\). Assume that it is true for \(n = r\). We need only to show that it is also true for \(n = r + 1\), that is, \(B_k^k\) divides \(B_{(r+1)k}^k\). Since \(B_{(r+1)k}^k = B_{rk+k}^k = B_{rk}^k C_k^k + C_{rk}^k B_k^k\) by Theorem 2.5, \((B_k^k, C_k^k) = 1\) by Lemma 2.9 and \(B_k^k\) divides \(B_{rk}^k\) by assumption, it follows that \(B_k^k\) divides \(B_{(r+1)k}^k\).

**Lemma 2.11:** If n and k are natural numbers, then \((B_k^k, C_{nk}^k) = 1\).

**Proof:** By Lemma 2.9, \((B_{nk}^k, C_{nk}^k) = 1\). Since \(B_k^k\) divides \(B_{nk}^k\) by Lemma 2.10, it follows that \((B_k^k, C_{nk}^k) = 1\).

**Lemma 2.12:** If n and k are natural numbers and \(B_k^k\) divides \(B_n^n\), then k divides n.

**Proof:** Certainly \(n \geq k\). If \(n = k\) then the proof is trivial. Assume that \(n > k\). Then by Euclid’s division lemma ([2], Theorem 2.1), there exists integers q and r such that \(q \geq 1, 0 \leq r < k\) and \(n = qk + r\). By Theorem 2.5, \(B_n^n = B_{qk+r}^q C_r + C_{qk}^q B_r\). Since \(B_k^k\) divides \(B_{qk}^q\) by Lemma 2.10, and \((B_k^k, C_{qk}^q) = 1\) by Lemma 2.11, it follows that \(B_k^k\) divides \(B_r^r\). Since \(r < k\), it follows that \(B_r^r = 0\) and hence \(r = 0\). Thus \(n = qk\) and therefore k divides n.

It can now be readily seen that Theorem 2.8 directly follows from Lemmas 2.10 and 2.12.

The following theorem tells something more than Theorem 2.8.

**Theorem 2.13:** If m and n are natural numbers, then \((B_m^m, B_n^n) = B_{(m, n)}\).

**Proof:** If \(m = n\), the proof is trivial; else let us assume without loss of generality that \(m < n\). By Euclid’s division lemma, there exists integers \(q_i\) and \(r_i\) such that \(q_i \geq 1, 0 \leq r_i < m\) and \(n = qm + r_i\). Now by Theorem 2.5
\[
(B_m^m, B_n^n) = (B_m^m, B_{qm+r_i}^i) = (B_m^m, B_{qm}^i C_{r_i} + C_{qm}^i B_{r_i}^i).
\]
Since $B_m$ divides $B_m^{q_m}$ by Lemma 2.10 and $(B_m, C_{q,m}) = 1$ by Lemma 2.11, it follows that $(B_m, B_n) = (B_m, B_n)$ and $(m, n) = (m, q_1 m + r_1) = (m, r_1)$. If $r_1 > 0$, then there exists integers $q_2$ and $r_2$ such that $q_2 \geq 1, 0 \leq r_2 < r_1$ and $m = q_2 r_1 + r_2$. Now again by Theorem 2.5,

$$(B_m, B_n) = (B_m, B_n)$$

$$= (B_{q_2 n + r_2}, B_n)$$

$$= (B_{q_2 n C_2 + C_{q_2 n} B_2}, B_n)$$

$$= (B_{r_2}, B_n),$$

and $(m, r_1) = (q_2 r_1 + r_2, r_1) = (r_2, r_1)$. This process may be continued till a newly arising $r_i$ does not equal to zero. Since $r_1 > r_2 > \cdots$, it follows that $r_i \leq m - i$, so that after at most $m$ steps some $r_i$ will be equal to zero. If $r_{k-1} > 0$ and $r_k = 0$, then we have

$$(B_m, B_n) = (B_{r_{k-2}}, B_{r_{k-1}}) = (B_{q_k r_{k-1}}, B_{r_{k-1}}) = B_{r_{k-1}}$$

and $(m, n) = (r_{k-2}, r_{k-1}) = (q_k r_{k-1}, r_{k-1}) = r_{k-1}$. Thus $(B_m, B_n) = B_{r_{k-1}} = B_{(m, n)}$ and the proof is complete.

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