Estimating Quantiles of Two Exponential Populations under Ordered Location Using Censored Samples

Adarsha Kumar Jena and Manas Ranjan Tripathy*

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Abstract. The problem of component wise estimation of quantiles of two shifted exponential populations has been considered under type-II censored samples when the location parameters assume certain ordering. When there is no order restriction on the location parameters, estimators like maximum likelihood estimator (MLE), modified maximum likelihood estimator (MMLE), uniformly minimum variance unbiased estimator (UMVUE) and best affine equivariant estimator (BAEE) have been found. Incorporating the ordered restriction on the location parameters, isotonic estimators of the BAEE and the mixed estimators have been obtained. Further, using prior information of ordered location parameters, certain Bayes estimators have been obtained. All the proposed estimators have been compared using Monte-Carlo simulation technique. Finally conclusions have been made regarding the use of the estimators.

*Keywords:*Bayes estimator; Best affine equivariant estimator (BAEE); Estimation of quantiles; Modified maximum likelihood estimator (MMLE); Mixed estimator; Order Restriction; Type-II censoring; Uniformly minimum variance unbiased estimator (UMVUE).

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December, 2017

Outline

Model

- Let $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(r)}$ $(2 \leq r \leq m)$ and $Y_{(1)} \leq Y_{(2)} \leq \ldots \leq Y_{(s)}$ $(2 \leq s \leq n)$ be the r and s number of ordered observations taken from two random samples of sizes $m(\geq 2)$ and $n(\geq 2)$, which follow exponential distributions with a common scale parameter σ and different location parameters μ_1 and μ_2 respectively.
- Since μ_is denote the minimum guarantee time, we assume $\mu_i \geq 0; i = 1, 2.$
- The problem is to estimate the p^{th} quantile $\theta_i = \mu_i + \eta\sigma$ of ith population, where $0 < \eta = -\log(1-p); 0 < p < 1.$ The loss function is taken as

$$L(d, \mu_1, \mu_2, \sigma) = \left(\frac{d - \theta_i}{\sigma}\right)^2 \tag{1}$$

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Introduction

- Let us consider a practical situation where our model fit well. Suppose a product/an equipment is produced from two different manufacturers, say M_1 and M_2. Let the life times of these products follow exponential distribution. Assume that both the manufactures employ modern statistical technique so that their variations will be minimized.
- Depending upon their technology development and the target level the minimum guarantee period or the mean life times of one manufacture will be less or more than the other. Under such a scenario it is quite practical to assume that the scale parameters are equal and location parameters are ordered.

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- Yike and Heliang (1999) have focused on the Bayesian estimation of ordered location parameters of two shifted exponential distributions using multiple type-II censoring scheme. But they have assumed that scale parameters are known.
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- Madi and Leonard (1996) have considered several shifted exponential populations with different location parameters. For their model they have taken scale parameters are equal. that is σ is the common scale parameter. They have found Bayesian estimation of σ under quadratic loss function without having restriction on location parameters.

Estimators without Order Restriction

- Define $X_{(1)} = \min_{1\leq j\leq m} X_j$; $Y_{(1)} = \min_{1\leq j\leq n} Y_j$; $T = T_1 + T_2$;

$$T_1 = \sum_{j=1}^{r} (X_j - X_{(1)}) + (m-r)(X_r - X_{(1)}),$$

$$T_2 = \sum_{j=1}^{s} (Y_j - Y_{(1)}) + (n-s)(Y_s - Y_{(1)}).$$

- $(X_{(1)}, Y_{(1)}, T)$ is a complete and sufficient statistic.
- $X_{(1)} \sim Exp(\mu_1, \sigma/m)$, $Y_{(1)} \sim Exp(\mu_2, \sigma/n)$ and $T \sim G(m+n-2, \sigma)$. (Elfessi and Pal(1991))

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- The MLEs for θ_1 is $X_{(1)} + \eta\frac{T}{r+s}$ and, for θ_2 is $Y_{(1)} + \eta\frac{T}{r+s}$.
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- Let $G_A = \{g_{a,b} : g_{a,b}(x) = ax + b, a \in \mathbb{R}^+, b \in \mathbb{R}\}$ be an affine group of transformations.
- The form of an affine equivariant estimator for the quantile θ_1 based on $(X_{(1)}, Y_{(1)}, T)$ is

$$d(X_{(1)}, Y_{(1)}, T) = X_{(1)} + c_1 T. \tag{2}$$

- $X_{(1)} + \frac{(\eta - \frac{1}{m})}{r+s-1} T$ is the BAEE for θ_1.
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Improved Estimators

- When there is no order restriction on θ_1 and θ_2, let $\hat{\theta}_1$ and $\hat{\theta}_2$ be some estimators for θ_1 and θ_2 respectively.
- But when $\theta_1 \leq \theta_2$, we take $\hat{\theta}_{1R} = \min(\hat{\theta}_1, \frac{r\hat{\theta}_1 + s\hat{\theta}_2}{r+s})$ and $\hat{\theta}_{2R} = \max(\hat{\theta}_2, \frac{r\hat{\theta}_1 + s\hat{\theta}_2}{r+s})$.
- Now for mixed estimator using BAEE, let us take $d_1 = X_{(1)} + c_1^* T$, $d_2 = Y_{(1)} + c_2^* T$, $c_1^* = \frac{(\eta - \frac{1}{m})}{r+s-1}$ and $c_2^* = \frac{(\eta - \frac{1}{n})}{r+s-1}$.
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Mixed Estimators

- The values of α^+ for which the risk of the mixed estimators have minimum risk is given by

$$\alpha^+ = \frac{-m}{2}\left[(\frac{\mu_2 - \mu_1}{\sigma}) + \frac{1}{m+n} - \eta + \frac{c_2^*(r+s-2)}{1+m(c_2^*-c_1^*)}\right] \quad (4)$$

- Inf $\alpha^+ = -\infty$ and Sup $\alpha^+ = \frac{-m}{2}\left[\frac{1}{m+n} - \eta + \frac{c_2^*(r+s-2)}{1+m(c_2^*-c_1^*)}\right]$.

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The mixed estimator $d_\alpha(\underline{d})$ is inadmissible for $\alpha^+ >$ Sup α^+ and is improved by $d_{sup\alpha^+}$ and, is admissible for $\alpha^+ \leq$ Sup α^+ among the class of estimators.

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Mixed Estimators for θ_2

- The mixed estimator for θ_2 is given by

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- $\pi_1(\mu_1, \mu_2) = c$ for $\mu_1 \leq \mu_2$, where c is a constant.
- $\pi_2(\sigma) = \frac{1}{\sigma}$ for $\sigma > 0$.
- let us denote the sufficient statistics $(X_{(1)}, Y_{(1)}, T)$ as (X, Y, T).
- The likelihood function is given by

$$L(x, y, t) = \frac{mnct^{r+s-3}}{\Gamma(r+s-2)\sigma^{r+s}} e^{-\frac{1}{\sigma}\{mx+ny+t-m\mu_1-n\mu_2\}} \quad (7)$$

- Since $x > \mu_1$ and $y > \mu_2$, we get $0 < \mu_1 < \min(x, y)$ and $\mu_1 < \mu_2 < y$. Let us denote $t^* = \min(x, y)$.

Bayes Estimators

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- Since $x > \mu_1$ and $y > \mu_2$, we get $0 < \mu_1 < \min(x, y)$ and $\mu_1 < \mu_2 < y$. Let us denote $t^* = \min(x, y)$.

- $\pi_1(\mu_1, \mu_2) = c$ for $\mu_1 \leq \mu_2$, where c is a constant.
- $\pi_2(\sigma) = \frac{1}{\sigma}$ for $\sigma > 0$.
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Bayes Estimators

- The joint posterior density of μ_1, μ_2 and σ is obtained by

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where $A = \int_0^{t^*} \int_{\mu_1}^{y} \int_0^{\infty} g(\mu_1, \mu_2, \sigma|\underline{Z}) d\sigma d\mu_2 d\mu_1$.

- The Bayesian estimation of θ_1 is given by $\hat{\theta}_{1bs} = \frac{E(\frac{\theta_1}{\sigma^2}|\underline{z})}{E(\frac{1}{\sigma^2}|\underline{z})}$, where $E(\frac{\theta_1}{\sigma^2}|\underline{z})$ and $E(\frac{1}{\sigma^2}|\underline{z})$ are posterior mean of $\frac{\theta_1}{\sigma^2}$ and $\frac{1}{\sigma^2}$ respectively.
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Bayes Estimators

- Denoting $\xi = mx + t$ and $w = mx + ny + t$, it is found that

$$E(\frac{\mu_1}{\sigma^2}|\underline{z}) = \frac{mct^{r+s-3}\Gamma(r+s+1)}{A\Gamma(r+s-2)}(B_1 - B_2), \qquad (10)$$

where $$B_1 = \frac{w\{w^{-(r+s)} - (w-(m+n)t^*)^{-(r+s)}\}}{(m+n)^2(r+s)} + \frac{\{w^{1-(r+s)} - (w-(m+n)t^*)^{1-(r+s)}\}}{(m+n)^2(1-(r+s))};$$

$$B_2 = \frac{\xi\{\xi^{-(r+s)} - (\xi - mt^*)^{-(r+s)}\}}{m^2(r+s)} + \frac{\{\xi^{1-(r+s)} - (\xi - mt^*)^{1-(r+s)}\}}{m^2(1-(r+s))}.$$

Bayes Estimators

- Similarly,

$$E(\frac{1}{\sigma}|\underline{z}) = \frac{mct^{r+s-3}\Gamma(r+s)}{(r+s-1)A\Gamma(r+s-2)}(D_1 - D_2), \tag{11}$$

where

$$D_1 = \frac{1}{(m+n)}\{w^{1-(r+s)} - (w-(m+n)t^*)^{1-(r+s)}\};$$

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- The Bayes estimator of θ_2 under the loss function(1) is obtained by
$$\hat{\theta}_{2bs} = \frac{\int_{\mu_1=0}^{t^*}\int_{\mu_2=\mu_1}^{y}\int_{\sigma=0}^{\infty}(\frac{\mu_2}{\sigma^2}+\eta\frac{1}{\sigma})d\sigma d\mu_2 d\mu_1}{\int_{\mu_1=0}^{t^*}\int_{\mu_2=\mu_1}^{y}\int_{\sigma=0}^{\infty}(\frac{1}{\sigma^2})d\sigma d\mu_2 d\mu_1} \quad (14)$$

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Bayes Estimators

$$E(\frac{\mu_2}{\sigma^2}|\underset{\sim}{z}) = \frac{mct^{r+s-3}\Gamma(r+s)}{nA\Gamma(r+s-2)}(B_1^* - B_2^*), \text{ where} \tag{15}$$

$$B_1^* = \frac{nw\{w^{-(r+s)} - (w-(m+n)t^*)^{-(r+s)}\}}{(m+n)^2} + \frac{(m+n(r+s+1))\{w^{1-(r+s)} - (w-(m+n)t^*)^{1-(r+s)}\}}{(m+n)^2(1-(r+s))};$$

$$B_2^* = \frac{ny}{m}\{\xi^{-(r+s)} - (\xi - mt^*)^{-(r+s)}\} + \frac{\{\xi^{1-(r+s)} - (\xi - mt^*)^{1-(r+s)}\}}{m(1-(r+s))}.$$

- Hence the Bayesian estimation of θ_2 is given by

$$\hat{\theta}_{2bs} = \frac{\frac{1}{n}(B_1^* - B_2^*) + \frac{\eta}{(r+s-1)}(D_1 - D_2)}{(E_1 - E_2)}. \tag{16}$$

Bayes Estimators with Inverse Gamma Prior

- $\pi(\mu_1, \mu_2, \sigma) = \pi_1(\mu_1|\mu_2, \sigma)\pi_2(\mu_2|\sigma)\pi_3(\sigma)$, where

$$\pi_1(\mu_1|\mu_2, \sigma) = \frac{1}{\sigma}e^{-(\mu_2-\mu_1)/\sigma}, \quad \pi_2(\mu_2|\sigma) = \frac{1}{\sigma}e^{-\mu_2/\sigma} \tag{17}$$

$$\text{and } \pi_3(\sigma) = \frac{\beta^\alpha}{\Gamma(\alpha)}\frac{e^{-\beta/\sigma}}{\sigma^{\alpha+1}}, \ \alpha > 0, \beta > 0. \tag{18}$$

- The joint posterior density of μ_1, μ_2 and σ is given by

$$g(\mu_1, \mu_2, \sigma|\underline{Z}) = \frac{\beta^\alpha mnt^{r+s-3}}{A\Gamma(\alpha)\Gamma(r+s-2)}\frac{1}{\sigma^{r+s+\alpha+3}} e^{-\frac{1}{\sigma}\{mx+ny+t+\beta+(2-n)\mu_2-(m+1)\mu_1\}}, \tag{19}$$

where $A = \int_0^{t^*}\int_{\mu_1}^{y}\int_0^\infty g(\mu_1, \mu_2, \sigma|\underline{Z})d\sigma d\mu_2 d\mu_1$.

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Bayes Estimators with Inverse Gamma Prior

- Denoting $v = mx + ny + t + \beta$ and $u = mx + 2y + t + \beta$, it is found that

$$E(\frac{\mu_1}{\sigma^2}|\underline{z}) = \frac{mnt^{r+s-3}\beta^{\alpha}\Gamma(r+s+3)}{A\Gamma(r+s-2)\Gamma(\alpha)(n-2)}(b_1 - b_2), \qquad (20)$$

where

$$b_1 = \frac{v\{v^{-(r+s+\alpha+2)} - (v-(m+n-1)t^*)^{-(r+s+\alpha+2)}\}}{(m+n-1)^2(r+s+\alpha+2)} - \frac{\{v^{-(r+s+\alpha+1)} - (v-(m+n-1)t^*)^{(r+s+\alpha+1)}\}}{(m+n-1)^2(r+s+\alpha+1)};$$

and

$$b_2 = \frac{u\{u^{-(r+s+\alpha+2)} - (u-(m+1)t^*)^{-(r+s+\alpha+2)}\}}{(m+1)^2(r+s+\alpha+2)} - \frac{\{u^{-(r+s+\alpha+1)} - (u-(m+1)t^*)^{(r+s+\alpha+1)}\}}{(m+1)^2(r+s+\alpha+1)}.$$

Bayes Estimators with Inverse Gamma Prior

- Similarly,

$$E(\frac{1}{\sigma}|\underline{z}) = \frac{mnt^{r+s-3}\beta^{\alpha}\Gamma(r+s+1)}{A\Gamma(r+s-2)\Gamma(\alpha)(n-2)}(d_1 - d_2), \qquad (21)$$

where

$$d_1 = \frac{1}{(m+n-1)}\{v^{-(r+s+\alpha+1)} - (v-(m+n-1)t^*)^{-(r+s+\alpha+1)}\};$$

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where

$$b_1^* = \frac{\{u^{-(r+s+\alpha+1)} - (u-(m+1)t^*)^{-(r+s+\alpha+1)}\}}{(m+n)(r+s+\alpha+1)} - \frac{(n-2)y}{m+1}\{u^{-(r+s+\alpha+2)} - (u-(m+1)t^*)^{-(r+s+\alpha+2)}\};$$

$$b_2^* = \frac{(m+1+(n-2)(r+s+\alpha+3))\{v^{-(r+s+\alpha+1)} - (v-(m+n-1)t^*)^{-(r+s+\alpha+1)}\}}{(m+n-1)^2(r+s+\alpha+1)} - \frac{(n-2)v}{(m+n-1)^2}\{v^{-(r+s+\alpha+2)} - (v-(m+n-1)t^*)^{-(r+s+\alpha+2)}\}.$$

- Hence the Bayesian estimation of θ_2 is obtained by

$$\hat{\theta}_{2bs} = \frac{(\frac{r+s+\alpha+1}{n-2})(b_1^* - b_2^*) + \eta(d_1 - d_2)}{(r+s+\alpha+1)(e_1 - e_2)}. \tag{25}$$

Simulation Results

- The percentage of relative risk improvent (PRRI) of any estimator δ_i w.r.t. the MLE is given by

$$R_i = (1 - \frac{\delta_i}{MLE}) \times 100.$$

- The PRRIs of all the estimators are very negligible except the the Bayes estimators.
- The PRRIs are highly dependent on the parameters α and β than the number of samples m and n. It may be positive or negative. However, when the parameters α and β are nearer to each other$(\alpha \approx \beta)$, the PRRI is noticeable.

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Table 1: For θ_1 with $\eta = 1.5; \alpha = 3.5; \beta = 3.0$; $C.F. = (.25, .5, .75, 1)$

μ_1/σ	μ_2/σ	$R(d_{BA})$	$R(d_R)$	$R(d_M)$	$R(d_{BC1})$	$R(d_{BC2})$
0.5	1.0	6.419	6.464	6.262	12.042	77.441
		1.283	1.432	1.270	5.211	56.872
		0.000	0.302	0.159	2.717	44.588
		0.884	1.334	1.237	3.347	37.930
1.0	2.5	5.578	5.578	5.578	8.686	88.519
		1.076	1.076	1.076	1.849	67.965
		0.000	0.000	0.000	0.161	52.599
		0.595	0.595	0.595	0.578	43.071
2.0	3.5	6.341	6.341	6.341	8.009	79.809
		1.082	1.082	1.082	1.522	56.747
		0.000	0.000	0.000	0.164	42.164
		0.631	0.631	0.631	0.716	35.622
2.5	4.5	5.906	5.906	5.906	6.994	51.828
		1.199	1.199	1.199	1.351	24.412
		0.000	0.000	0.000	0.043	15.388
		0.445	0.445	0.445	0.462	10.804

Table 2: For θ_2 with $\eta=1.5;\alpha=3.5;\beta=3.0$; $C.F.=(.25,.5,.75,1)$

μ_1/σ	μ_2/σ	$R(d_{BA})$	$R(d_R)$	$R(d_M)$	$R(d_{BC1})$	$R(d_{BC2})$
0.5	1.0	7.0870	7.149	7.267	12.719	77.031
		1.056	1.116	1.112	4.740	54.778
		0.000	0.123	0.150	2.205	41.581
		0.710	0.781	0.779	1.892	33.405
1.0	2.5	5.218	5.218	5.218	8.736	85.504
		0.963	0.963	0.963	2.245	61.560
		0.000	0.000	0.000	0.566	45.233
		0.779	0.779	0.779	1.116	35.411
2.0	3.5	6.372	6.374	6.379	7.949	74.854
		1.386	1.386	1.386	1.628	50.288
		0.000	0.000	0.000	0.037	34.260
		0.681	0.681	0.681	0.670	25.079
2.5	4.5	6.268	6.268	6.268	7.303	44.235
		0.928	0.928	0.928	1.046	13.806
		0.000	0.000	0.000	0.013	2.098
		0.188	0.188	0.188	0.186	0.574

Conclusion

- The performance of the BAEE, the mixed estimators and the restricted BAEE are almost same.
- The Bayes estimators perform better than other estimators.
- The performance of Bayes estimators decrease as the censoring factors increase when μ_1/σ and μ_2/σ are nearer to each other.
- When μ_1/σ and μ_2/σ are far from each other, the performance of Bayes estimators are not better than the other estimators.

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Thank You