ALMOST BALANCING-LIKE SEQUENCES

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Introduction

The class of linear binary recurrences

$$x_{n+1} = Ax_n - x_{n-1}$$

with initial terms $x_0 = 0, x_1 = 1$ are important in the sense that the case $A = 2$, refers to natural numbers. The sequence of balancing numbers is a special case of this class of this class corresponding to $A = 6$. The sequences generated from this class of binary recurrences are all strong divisibility sequence and satisfy many properties resembling those of balancing numbers. This class of sequence with $A > 2$ are known as balancing-like sequences and like natural numbers, each balancing-like sequence admits the definition of triangular-like numbers.

The n^{th} triangular number is equal to the sum of first $\boldsymbol{n}$ natural numbers and is also half the product of $\boldsymbol{n}$ and $\boldsymbol{n+1}$. However, in case of balancing-like numbers, the n^{th} triangular-like number is defined as the product of the n^{th} and $(n+1)^{\text{st}}$ balancing-like numbers divided by $\boldsymbol{A}$. But unlike natural numbers, the difference of n^{th} and $(n-1)^{\text{st}}$ triangular-like numbers is not equal to the n^{th} term of the same balancing-like sequence, rather, it is equal to the n^{th} term of another balancing-like sequence defined by

$$\boldsymbol{x_{n+1} = (A^2 - 2)x_n - x_{n-1}}$$

with initial terms $\boldsymbol{x_0 = 0, x_1 = 1}$. There are numerous mysteries and possibilities associated with the balancing-like and associated sequences. This presentation is devoted to the study of the almost balancing-like sequences associated with the balancing-like sequences.

Balancing Numbers

Balancing numbers $\boldsymbol{B}$ and balancers $\boldsymbol{R}$ are solutions of the Diophantine equation

$$\mathbf{1} + \mathbf{2} + \cdots + (\boldsymbol{B} - \mathbf{1}) = (\boldsymbol{B} + \mathbf{1}) + (\boldsymbol{B} + \mathbf{2}) + \cdots + (\boldsymbol{B} + \boldsymbol{R}).$$

Thus, $\mathbf{6}, \mathbf{35}$ and $\mathbf{204}$ are the first three balancing numbers with balancers $\mathbf{2}, \mathbf{14}$ and $\mathbf{84}$ respectively.

The definition of balancing numbers is due to Behera and Panda: *On the square roots of triangular numbers*, Fib. Quart., 37(1999), 98-205.

The concept of balancing numbers also coincides with the concept of numerical centers described in a paper by R. Finkelstein, *The house problem*, Amer. Math. Monthly, 72, 1965.

A natural number n is a balancing number if and only if $\boldsymbol{8n^2+1}$ is a perfect square.

$$\mathbf{8\cdot 6^2+1=17^2}$$

$$\mathbf{8\cdot 35^2+1=99^2}$$

$$\mathbf{8\cdot 204^2+1=577^2}$$

Since

$$\mathbf{8\cdot 1^2+1=3^2},$$

1 is accepted as the first balancing number, though it does not satisfy the defining equation of balancing numbers.

If $\boldsymbol{B}$ is a balancing number, then the next one is

$$\mathbf{3}\boldsymbol{B}+\sqrt{\mathbf{8}\boldsymbol{B}^{\mathbf{2}}+\mathbf{1}}$$

and the previous one is

$$\mathbf{3}\boldsymbol{B}-\sqrt{\mathbf{8}\boldsymbol{B}^{\mathbf{2}}+\mathbf{1}}\ .$$

The $\boldsymbol{n}^{\boldsymbol{th}}$ balancing number is denoted by $\boldsymbol{B}_{\boldsymbol{n}}$.

The number

$$\boldsymbol{C}_{\boldsymbol{n}}=\sqrt{\mathbf{8}\boldsymbol{B}_{\boldsymbol{n}}{}^{\mathbf{2}}+\mathbf{1}}$$

is called the $\boldsymbol{n}^{\boldsymbol{th}}$ Lucas-balancing number.

Balancing numbers satisfy

$$\boldsymbol{B_{n+1} = 6B_n - B_{n-1}\,;\;\; B_1 = 1, B_2 = 6.}$$

Lucas-balancing numbers satisfy

$$\boldsymbol{C_{n+1} = 6C_n - C_{n-1}\,;\;\; C_1 = 3, B_2 = 17.}$$

Their Binet forms are

$$\boldsymbol{B_n = \frac{{\lambda_1}^n - {\lambda_2}^n}{4\sqrt{2}}}\,,$$

and

$$\boldsymbol{C_n = \frac{{\lambda_1}^n + {\lambda_2}^n}{2}}\,,$$

respectively, where $\boldsymbol{\lambda_1 = 3 + 2\sqrt{2}}$, $\boldsymbol{\lambda_2 = 3 - 2\sqrt{2}}$.

Recurrence Relation (Non-linear)

$$\boldsymbol{B_n}^2 = \mathbf{1} + \boldsymbol{B_{n-1}} \cdot \boldsymbol{B_{n+1}}$$

Equivalently,

$$(\boldsymbol{B_n} - \mathbf{1})(\boldsymbol{B_n} + \mathbf{1}) = \boldsymbol{B_{n-1}} \cdot \boldsymbol{B_{n+1}}$$

which is looking more natural. In general, the balancing numbers satisfy

$$\boldsymbol{B_{m-n}} \cdot \boldsymbol{B_{m+n}} = (\boldsymbol{B_m} - \boldsymbol{B_n})(\boldsymbol{B_m} + \boldsymbol{B_n}).$$
$$= \boldsymbol{B_m}^2 - \boldsymbol{B_n}^2$$

The Fibonacci numbers satisfy

$$\boldsymbol{F_{m-n}} \cdot \boldsymbol{F_{m+n}} = \boldsymbol{F_m}^2 - (-\mathbf{1})^{\boldsymbol{m+n}} \boldsymbol{F_n}^2$$

which is not so natural like the one for balancing numbers.

Balancing numbers behave like natural numbers:

$$\boldsymbol{B_1 + B_3 + \cdots + B_{2n-1} = {B_n}^2}$$

$$\boldsymbol{B_2 + B_4 + \cdots + B_{2n} = B_n \cdot B_{n+1}}.$$

The Fibonacci numbers do not have such type of properties.

De-Moivre's theorem for balancing numbers:

$$\boldsymbol{\left(C_1 \pm \sqrt{8}B_1\right)^n = C_n \pm \sqrt{8}B_n}$$

More generally,

$$\boldsymbol{\left(C_m \pm \sqrt{8}B_m\right)^n = C_{mn} \pm \sqrt{8}B_{mn}}$$

Also,

$$\boldsymbol{\left(C_m \pm \sqrt{8}B_m\right)\left(C_n \pm \sqrt{8}B_n\right) = \left(C_mC_n \pm \sqrt{8}B_mB_n\right)}$$

The identity

$$\boldsymbol{B_{m\pm n} = B_m C_n \pm C_m B_n}$$

resembles the trigonometric identity

$$\mathbf{sin}\,(\boldsymbol{x} \pm \boldsymbol{y}) = \mathbf{sin}\,\boldsymbol{x}\,\mathbf{cos}\,\boldsymbol{y} \pm \mathbf{cos}\,\boldsymbol{x}\,\mathbf{sin}\,\boldsymbol{y}$$

The corresponding Fibonacci identity is

$$\boldsymbol{F_{m\pm n} = \frac{1}{2}[F_m L_n \pm L_m F_n].}$$

Like Fibonacci numbers, the sequence of balancing numbers is a strong divisibility sequence.

$$\boldsymbol{(B_m\,, B_n) = B_{(m\,,n)}}$$

where $(m\,, n)$ denotes the greatest common divisor of m and n.

Balancing–like Sequences

A balancing-like sequence is a recurrence sequence defined as

$$x_{n+1} = Ax_n - x_{n-1}\,; \;\; x_0 = 0, x_1 = 1$$

where $A > 2$ is any natural number. If $A = 6$, this sequence coincides with the sequence of balancing numbers. Each balancing-like sequence satisfies all important properties of the balancing numbers. However, balancing-like sequences do not have defining Diophantine equations like balancing numbers.

Sum formulas

$$x_1 + x_3 + \cdots + x_{2n-1} = x_n{}^2$$

$$x_2 + x_4 + \cdots + x_{2n} = x_n \cdot x_{n+1}$$

If $A^2 - 4$ is not a perfect square then

$$x_{m+n} = x_m y_n + y_m x_n$$

where

$$y_n = \sqrt{D x_n{}^2 + 1}, \qquad D = \frac{A^2 - 4}{4}$$

The Lucas-balancing-like sequence $\{\boldsymbol{y_n}\}$ associated with the balancing-like sequence $\{\boldsymbol{x_n}\}$ defined as

$$\boldsymbol{y_n = \sqrt{D{x_n}^2 + 1}, \qquad D = \frac{A^2 - 4}{4}}$$

is an integer sequence if n is even and satisfies the recurrence relation

$$\boldsymbol{y_{n+1} = Ay_n - y_{n-1}\,;\;\; y_0 = 1,\;\; y_1 = \frac{A}{2}}$$

identical with the recurrence relation of the sequence $\{\boldsymbol{x_n}\}$.

Like the Fibonacci and balancing sequence, each balancing-like sequence satisfies the strong divisibility property

$$(x_m, x_n) = x_{(m,n)}$$

where (m, n) denotes the greatest common divisor of m and n.

In addition,

$$x_{m-n} \cdot x_{m+n} = (x_m - x_n)(x_m + x_n)$$

Also, the De-Moivre's theorem holds.

$$\left(y_1 \pm \sqrt{D} x_1\right)^n = y_n \pm \sqrt{D} x_n$$

More generally,

$$\left(y_m \pm \sqrt{D} x_m\right)^n = y_{mn} \pm \sqrt{D} x_{mn}.$$

Also,

$$\left(y_m \pm \sqrt{D} x_m\right)\left(y_n \pm \sqrt{D} x_n\right) = \left(y_m y_n \pm \sqrt{D} x_m x_n\right).$$

Almost Balancing Sequences

A natural number n is called an **almost balancing number** if it satisfies the Diophantine equation

$$|\{(n+1)+(n+2)+\cdots+(n+r)\}$$

If $\{(n+1)+(n+2)+\cdots+(n+r)\}-\{1+2+\cdots+(n-1)\}=1,$

we call n an **almost balancing number of first kind (**A_1-*balancing numbers*) while if

$$\{(n+1)+(n+2)+\cdots+(n+r)\}-\{1+2+\cdots+(n-1)\}=-1$$

we call n an **almost balancing number of second kind (**A_2-*balancing numbers*). In the former case, we call r an **almost balancer of first kind**, while in the latter case we call r an **almost balancer of second kind**.

If $\boldsymbol{x}$ is an A_1-balancing number, then $\mathbf{8x^2+9}$ is a perfect square while if $\boldsymbol{x}$ is an A_2-balancing number then $\mathbf{8x^2-7}$ is a perfect square. Using the theory of Pell's equation, it is easy to see that the A_1-balancing numbers are given by

$$\boldsymbol{u_n = 3B_n, n = 1, 2, \cdots}$$

while the A_2-balancing numbers partition in following two classes:

$$\boldsymbol{v_n = B_n - 2B_{n-1}}$$

and

$$\boldsymbol{w_n = 2B_n - B_{n-1}, n = 1, 2, \cdots.}$$

Further relationships

- If x is a balancing number then $\alpha(x) = -5x + 2\sqrt{8x^2+1}$ and $\beta(x) = -x + \sqrt{8x^2+1}$ are A_2-balancing numbers.
- If x is an A_2-balancing number then either $\frac{\sqrt{8x^2-7}+x}{7}$ or $\frac{\sqrt{8x^2-7}-x}{7}$ is a balancing number.
- If *x* is an A_2-balancing number then either $f(x) = \frac{5x+2\sqrt{8x^2-7}}{7}$ *or* $g(x) = \frac{x+\sqrt{8x^2-7}}{7}$ is a balancing number.
- If x is an A_2-balancing number then either $\frac{11x+3\sqrt{8x^2-7}}{7}$ or $\frac{9x+2\sqrt{8x^2-7}}{7}$ is the A_2-balancing number next to x.

Almost Balancing-like Numbers

x is a **balancing number** iff

$$1+2+\cdots+(x-1)=(x+1)+\cdots+(x+r)$$

x is an **almost balancing number** iff

$$1+2+\cdots+(x-1)\pm 1=(x+1)+\cdots+(x+r)$$

Also,

x is a **balancing number** iff

$$8x^2+1=\square,\ a\ perfect\ square$$

x is an **almost balancing number** iff

$$8\left(x^2\pm 1\right)+1=\square,\ a\ perfect\ square$$

The balancing numbers B_n can be recurrently defined as

$$\boldsymbol{x_{n+1} = 6x_n - x_{n-1}\,;\;\; x_0 = 0, x_1 = 1}$$

The **balancing-like numbers** x_n are defined as (for fixed $A > 2$)

$$\boldsymbol{x_{n+1} = Ax_n - x_{n-1}\,;\;\; x_0 = 0, x_1 = 1}$$

Also, $\boldsymbol{x}$ is a balancing number iff

$$\boldsymbol{8x^2 + 1 = \square}, a\ perfect\ square$$

$\boldsymbol{x}$ is a balancing-like number iff

$$\boldsymbol{Dx^2 + 1 = \square}, a\ perfect\ square$$

with $\boldsymbol{D = \frac{A^2-4}{4}}$.

Generalization from almost balancing to almost balancing-like numbers:

$\boldsymbol{x}$ is an *almost balancing number* iff

$$8(x^2 \pm 1) + 1 = \square, \text{ a perfect square}$$

This suggests us to define

$\boldsymbol{x}$ is an *almost balancing-like number* iff

$$D(x^2 \pm 1) + 1 = \square, \text{ a perfect square}$$

Definition: For a fixed natural number $\boldsymbol{A > 2}$, we call a natural number $\boldsymbol{x}$ an *almost balancing-like number* if and only if $\boldsymbol{D\left(x^2 \pm 1\right) + 1}$ is a perfect rational square. Similar to the case of almost balancing numbers, we call x an A_1-*almost balancing-like number* if $\boldsymbol{D(\ x^2 +}$

Case I: $\boldsymbol{A}$ **is even.** In this case $\boldsymbol{D}$ is a natural number. If $\boldsymbol{x}$ is an A_1-balancing-like number then $\boldsymbol{Dx^2+D+1}$ is a perfect square say $\boldsymbol{Dx^2+D+1=y^2}$ and this equation can be rewritten as

$$\boldsymbol{y^2-Dx^2=D+1=\frac{A^2}{4}}$$

which is a generalized Pell's equation. Similarly, if x is an A_2-balancing-like number then $\boldsymbol{Dx^2-D+1=y^2}$ can be rewritten as the generalized Pell's equation

$$\boldsymbol{y^2-Dx^2=-D+1}.$$

Case II: $\boldsymbol{A}$ is odd. In this case $\boldsymbol{D}$ is a rational number, an integral multiple of ½. If $\boldsymbol{x}$ is an A_1-balancing-like number then $\mathbf{4}\big(\boldsymbol{D}\boldsymbol{x}^{\mathbf{2}} +$

Almost balancing-like sequence corresponding to $A = 3$

The balancing-like sequence corresponding to $A = 3$ is the solution of

$$\boldsymbol{x_{n+1} = 3x_n - x_{n-1}\,;\;\; x_0 = 0, x_1 = 1}$$

and x is a member of this sequence if and only if $5x^2 + 4 = y^2$ for some $y \in \mathbb{Z}^+$. Thus, $\boldsymbol{x_n = F_{2n}}$. The A_1-balancing-like numbers corresponding to $A = 3$ are values of x satisfying the generalized Pell's equation $\boldsymbol{y^2 - 5x^2 = 9}$ and are of the form $\boldsymbol{\frac{3F_{6n}}{2}}$. Similarly, the A_2-balancing-like numbers are values of x satisfying the generalized Pell's equations $\boldsymbol{y^2 - 5x^2 = -1}$ and are of the form $\boldsymbol{\frac{F_{6n-3}}{2}}$.

Almost balancing-like sequence corresponding to $A = 3$

Expressed as numerical sequences, the balancing-like sequence corresponding to $A = 3$ is

$$\mathbf{1, 3, 8, 21, 55, 144, \ldots,}$$

the A_1- almost balancing-like sequence is

$$\mathbf{12, 216, 3876, 69552, 1248060, \ldots,}$$

and finally the A_2- almost balancing-like sequence is

$$\mathbf{1, 17, 305, 5473, 98209, 1762289, \ldots\ .}$$

Almost balancing-like sequence corresponding to $A = 3$

A_1-balancing-like numbers corresponding to $\boldsymbol{A = 3}$ satisfies the recurrence relation

$$\boldsymbol{u_{n+1} = 18u_n - u_{n-1}, u_0 = 0, u_1 = 12}$$

while the A_1-balancing-like numbers satisfies the recurrence relation

$$\boldsymbol{v_{n+1} = 18v_n - v_{n-1},\ s_0 = 1, s_1 = 17.}$$

Almost balancing-like sequence corresponding to $A = 4$

The balancing-like sequence corresponding to $\boldsymbol{A = 4}$ is the solution of

$$x_{n+1} = 4x_n - x_{n-1}\,; \;\; x_0 = 0, x_1 = 1$$

and x is a member of this sequence if and only if $\boldsymbol{3x^2 + 1 = y^2}$ for some $y \in \mathbb{Z}^+$. Thus, the A_1- and A_2-balancing-like numbers numbers corresponding to $\boldsymbol{A = 4}$ are values of x satisfying the generalized Pell's equations

$$y^2 - 3x^2 = 4 \text{ and } y^2 - 3x^2 = -2$$

respectively. Finally, the A_1- and the A_2-balancing-like numbers corresponding to $\boldsymbol{A = 4}$ are

$$u_n = 2x_n, n = 1, 2, \dots \text{ and } v_n = x_n - x_{n-1}, n = 1, 2, \dots$$

respectively.

Almost balancing-like sequence corresponding to $A = 5$

The balancing-like sequence corresponding to $A = 5$ is the solution of

$$x_{n+1} = 5x_n - x_{n-1}\,; \;\; x_0 = 0, x_1 = 1$$

and x is a member of this sequence if and only if $21x^2 + 4 = y^2$ for some $y \in \mathbb{Z}^+$. Thus, the A_1-balancing-like numbers corresponding to $A = 5$ are solutions in x of the generalized Pell's equation $y^2 - 21x^2 = 25$. These numbers partition in three classes and are given by

$$u_n = \frac{8x_{3n+1} - x_{3n}}{2}, n = 0, 1, 2, \dots$$

$$u'_n = \frac{16x_{3n+1} - 3x_{3n}}{2}, n = 0, 1, 2, \dots$$

$$u''_n = \frac{5x_{3n}}{2}, n = 1, 2, \dots$$

Almost balancing-like sequence corresponding to $A = 5$

The A_2-balancing-like numbers numbers corresponding to $\boldsymbol{A = 5}$ are solutions in $\boldsymbol{x}$ of the generalized Pell's equations

$$\boldsymbol{y^2 - 21x^2 = -17}$$

In this case, the A_2- balancing-like numbers partition in two classes and are of the form

$$\boldsymbol{v_n = \frac{2x_{3n+1} - 3x_{3n}}{2}, n = 0, 1, 2, \ldots}$$

and

$$\boldsymbol{v'_n = \frac{62x_{3n+1} - 13x_{3n}}{2}, n = 0, 1, 2, \ldots}$$

respectively.

Almost balancing-like sequence corresponding to $A = 6$

The balancing-like sequence corresponding to $\boldsymbol{A = 6}$ is nothing but the sequence of balancing numbers and hence the corresponding almost balancing-like sequence is the sequence of almost balancing numbers. The A_1-balancing numbers $\boldsymbol{x}$ are solutions of the generalized Pell's equation $\boldsymbol{y^2 - 8x^2 = 9}$, while the A_2-balancing numbers are values of $\boldsymbol{x}$ satisfying $\boldsymbol{y^2 - 8x^2 = -7}$. Further, A_1-almost balancing numbers are thrice the corresponding balancing numbers while, the A_2-almost balancing numbers partition into two classes and are of the form $\boldsymbol{2B_n - B_{n-1}}$ and $\boldsymbol{B_n - 2B_{n-1}, n = 1, 2, \ldots}$.

Almost balancing-like sequence corresponding arbitrary AThe

almost balancing-like numbers x corresponding to an even A (> 2) are solutions of the generalized Pell's equations

$$\boldsymbol{y^2 - Dx^2 = D + 1} \text{ and } \boldsymbol{y^2 - Dx^2 = -D + 1} \ (D = (A^2 - 4)/4)$$

the former being associated with A_1-balancing-like numbers, while the latter with A_2-balancing-like numbers. Parametric generalized Pell's equations cannot be solved completely in general; however, it is possible to extract few subclasses of A_1- almost balancing-like numbers.

Almost balancing-like sequence corresponding arbitrary A

For $\boldsymbol{A} = \boldsymbol{4}$ or $\boldsymbol{6}$, there is a subclass consisting of A_1-balancing-like numbers that are multiples of the corresponding balancing-like numbers. Such a class of almost balancing-like sequence is associated with even integral values $\boldsymbol{A} > \boldsymbol{2}$. For each $\boldsymbol{A}$, this subclass of A_1-balancing-like numbers that are given by

$$\boldsymbol{u_n = \frac{Ax_{3n}}{2}}, n = 1,2, \ldots .$$

When $\boldsymbol{A}$ is odd, this class of numbers multiple of $\boldsymbol{A}$, while if $\boldsymbol{A}$ is even, this class is a multiple of $\boldsymbol{A/2}$.

THANK YOU

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ANY QUESTION?