S-fibrations and Calculus Left Fractions

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Abstract

Let $\mathcal{C}$ be any small $\mathcal{U}$-category, where $\mathcal{U}$ is a fixed Grothendeick universe. Let S be a set of morphisms in the category $\mathcal{C}$. Let $\mathcal{C}[S^{-1}]$ be the category of fractions of S and $F_S : \mathcal{C} \to \mathcal{C}[S^{-1}]$ be the canonical functor. For convenience we write $F_S = F$. Bauer and Dugundji [2] have introduced the concept of S-fibration, weak S-fibration, S-cofibration and weak S-cofibration in the category $\mathcal{C}$ and have explored the properties of these concepts. There are some other advantages over the assumption that the set of morphisms S admits a calculus of left (right) fractions [4, 6]. In this note we study some cases showing how the assumption that S admits a calculus of left (right) fractions helps us to prove that weak S-fibration implies S-fibration and weak S-cofibration implies S-cofibration.

1. Calculus of left (right) fractions

The concepts of calculus of left fractions and right fraction play a crucial role in constructing the category of fractions $\mathcal{C}[S^{-1}]$.

1.1 Definition. ([6], p. 258) A family of morphisms S in the category $\mathcal{C}$ is said to admit a *calculus of left fractions* if

(a) S is closed under finite compositions and contains identities of $\mathcal{C}$,

(b) any diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ f\downarrow & & \\ Z & & \end{array}$$

in $\mathcal{C}$ with $s \in S$ can be completed to a diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ f\downarrow & & \downarrow g \\ Z & \xrightarrow[t]{} & W \end{array}$$

with $t \in S$ and $tf = gs$,

(c) given

$$X \xrightarrow{s} Y \underset{g}{\overset{f}{\rightrightarrows}} Z \overset{t}{\dashrightarrow} W$$

with $s \in S$ and $fs = gs$, there is a morphism $t : Z \to W$ in S such that $tf = tg$.

A simple characterization for a family of morphisms S to admit a calculus of left fractions is the following.

1.2 Theorem. ([3], Theorem 1.3, p. 67) *Let S be a closed family of morphisms of $\mathcal{C}$ satisfying*

(a) *if $uv \in S$ and $v \in S$, then $u \in S$,*

(b) *every diagram*

$$\begin{array}{ccc} \bullet & \xrightarrow{s} & \bullet \\ f\downarrow & & \\ \bullet & & \end{array}$$

in $\mathcal{C}$ with $s \in S$ can be embedded in a weak push-out diagram

$$\begin{array}{ccc} \bullet & \overset{s}{\rightarrow} & \bullet \\ f \downarrow & & \downarrow g \\ \bullet & \underset{t}{\rightarrow} & \bullet \end{array}$$

with $t \in S$.

Then S *admits a calculus of left fractions.*

The notion of a set of morphisms admitting a calculus of right fractions is defined dually.

1.3 Definition. ([6], p. 267) A family S of morphisms in a category $\mathcal{C}$ is said to admit a *calculus of right fractions* if

(a) any diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow f \\ Z & \underset{s}{\rightarrow} & Y \end{array}$$

in $\mathcal{C}$ with $s \in S$ can be completed to a diagram

$$\begin{array}{ccc} W & \overset{t}{\rightarrow} & X \\ g \downarrow & & \downarrow f \\ Z & \underset{s}{\rightarrow} & Y \end{array}$$

with $t \in S$ and $ft = sg$,

(b) given

$$W \overset{t}{\dashrightarrow} X \underset{g}{\overset{f}{\rightrightarrows}} Y \overset{s}{\rightarrow} Z$$

with $s \in S$ and $sf = sg$, there is a morphism $t : W \to X$ in S such that $ft = gt$.

The analog of Theorem 1.2 follows immediately by duality.

1.4 Theorem. ([3], Theorem1.3*, p. 70) *Let S be a closed family of morphisms of $\mathcal{C}$ satisfying*

(a) *if $vu \in S$ and $v \in S$, then $u \in S$,*

(b) *any diagram*

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow f \\ \bullet & \underset{s}{\rightarrow} & \bullet \end{array}$$

in $\mathcal{C}$ with $s \in S$, can be embedded in a weak pull-back diagram

$$\begin{array}{ccc} \bullet & \overset{t}{\rightarrow} & \bullet \\ g \downarrow & & \downarrow f \\ \bullet & \underset{s}{\rightarrow} & \bullet \end{array}$$

with $t \in S$.

Then S admits a calculus of right fractions.

We recall the definitions of Adams completion and cocompletion.

1.5. Definition. [4] Let $\mathcal{C}$ be an arbitrary category and S a set of morphisms of $\mathcal{C}$. Let $\mathcal{C}[S^{-1}]$ denote the category of fractions of $\mathcal{C}$ with respect to S and $F : \mathcal{C} \to \mathcal{C}[S^{-1}]$ be the canonical functor. Let $\mathcal{S}$ denote the category of sets and functions. Then for a given object Y of $\mathcal{C}$, $\mathcal{C}[S^{-1}](-, Y) : \mathcal{C} \to \mathcal{S}$ defines a contravariant functor. If this functor is representable by an object Y_S of $\mathcal{C}$, i.e., $\mathcal{C}[S^{-1}](-, Y) \cong \mathcal{C}(-, Y_S)$ then Y_S is called the (*generalized*) *Adams completion of* Y with respect to the set of morphisms S or simply the S-*completion* of Y. We shall often refer to Y_S as the *completion* of Y [4].

The above definition can be dualized as follows:

1.6. Definition. [3] Let $\mathcal{C}$ be an arbitrary category and S a set of morphisms of $\mathcal{C}$. Let $\mathcal{C}[S^{-1}]$ denote the category of fractions of $\mathcal{C}$ with respect S and $F: \mathcal{C} \to \mathcal{C}[S^{-1}]$ be the canonical functor. Let $\mathcal{S}$ denote the category of sets and functions. Then for a given object Y of $\mathcal{C}$, $\mathcal{C}[S^{-1}](Y, -) : \mathcal{C} \to \mathcal{S}$ defines a covariant functor. If this functor is representable by an object Y_S of $\mathcal{C}$, i.e., $\mathcal{C}[S^{-1}](Y, -) \cong \mathcal{C}(Y_S, -)$ then Y_S is called the (*generalized*) *Adams cocompletion* of Y with respect to the set of morphisms S or simply the S-*cocompletion of* Y. We shall often refer to Y_S as the cocompletion of Y [3].

The following results will be used in the sequel.

1.7 Theorem. ([3], Theorem 2.10, p. 76) *Let* S *be a saturated family of morphisms of the category* $\mathcal{C}$. *Then the following three statements are equivalent*:

(a) *Every object* Y *in* $\mathcal{C}$ *admits an* S-*completion*.

(b) S *admits a calculus of left fractions,* $\varinjlim P_Y$ *exists for all* Y, *where* $P_Y : \mathcal{C}(Y; S) \to \mathcal{C}$, *and* F_S *commutes with* $\varinjlim P_Y$.

(c) S *admits a calculus of left fractions,* $\varinjlim P_Y$ *exists for all* Y *and* F_S *commutes with all colimits in* $\mathcal{C}$.

1.8 Theorem. ([6], Lemma 19.2.6, p. 261) *Let* $\mathcal{C}$ *be an arbitrary category and* S *a set of morphisms of* $\mathcal{C}$. *Let* $\mathcal{C}[S^{-1}]$ *denote the category of fractions of* $\mathcal{C}$ *with respect to* S *and* $F_S : \mathcal{C} \to \mathcal{C}[S^{-1}]$ *be the canonical functor. Let the following hold*:

(a) S *consists of monomorphisms.*

(b) S *admits a calculus of left fractions.*

Then F_S *is faithful.*

2. S-fibrations

Each class S of morphisms in a category $\mathcal{C}$ determines a concept of fibration (and cofibration) in $\mathcal{C}$. We recall the concepts of S-fibration and weak S-fibration from [2].

2.1 Definition. [2] Let S be a subset of morphisms of $\mathcal{C}$. A morphism $p : E \longrightarrow B$ in $\mathcal{C}$ is called an S-*fibration* [2] if for each diagram

$$\begin{array}{ccccc} W & \overset{s}{\rightarrow} & X & \overset{g}{\rightarrow} & E \\ & & f \searrow & & \downarrow p \\ & & & & B \end{array}$$

with $s \in S$ and $pgs = fs$, there exists a morphism $g' : X \rightarrow E$ in $\mathcal{C}$

$$\begin{array}{ccccc} W & \overset{s}{\rightarrow} & X & \overset{g'}{\dashrightarrow} \underset{g}{\rightarrow} & E \\ & & f \searrow & & \downarrow p \\ & & & & B \end{array}$$

such that $gs = g's$ and $pg' = f$.

2.2 Definition. [2] Let S be a subset of morphisms of $\mathcal{C}$. A morphism $p : E \longrightarrow B$ in C is called a *weak* S-*fibration* [2] if for each diagram

$$\begin{array}{ccccc} W & \overset{s}{\rightarrow} & X & \overset{g}{\rightarrow} & E \\ & & f \searrow & & \downarrow p \\ & & & & B \end{array}$$

with $s \in S$ and $pgs = fs$, there exists a morphism $g' : X \to E$ in $\mathcal{C}$ and a morphism $t : X \to X$ with $t \in S$

$$\begin{array}{ccccccc} W & \overset{s}{\to} & X & \overset{t}{\to} & X & \begin{array}{c} g' \\ \dashrightarrow \\ \to \\ g \end{array} & E \\ & & & & & f \searrow & \downarrow p \\ & & & & & & B \end{array}$$

such that $gs = g's$, $ts = s$ and $pg' = ft$.

The following result is elementary in nature.

2.3 Proposition. *S-fibration implies weak S-fibration.*

Proof: Let $p : E \to B$ be an S-fibration in the category $\mathcal{C}$. In order to show that $p : E \to B$ is also a weak S-fibration consider an arbitrary diagram

$$\begin{array}{cccc} W & \overset{s}{\to} & X & \overset{g}{\to} \quad E \\ & & f \searrow & \downarrow p \\ & & & B \end{array}$$

with $s \in S$ and $pgs = fs$. Since $p : E \to B$ is a S-fibration, there exists a morphism $g' : X \to E$ in $\mathcal{C}$,

$$\begin{array}{ccccc} W & \overset{s}{\to} & X & \begin{array}{c} g' \\ \dashrightarrow \\ \to \\ g \end{array} & E \\ & & & f \searrow & \downarrow p \\ & & & & B \end{array}$$

such that $gs = g's$ and $pg' = f$. Considering $t = 1_X : X \to X$, we can have $gs = g's$ and $pg' = f1_X = ft$. This completes the proof of the Proposition 2.3. ■

Under some moderate assumptions on the set S, it can be proved that weak S-fibration always implies S-fibration.

2.4 Proposition. *Let S be the set of morphisms in $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{C}[S^{-1}]$ be the canonical functor. Suppose the following conditions hold*:

(a) $p : E \to B$ *is a weak S-fibration*.

(b) S *admits a calculus of left fractions*.

(c) S consists of monomorphisms.

Then $p : E \to B$ is an S-fibration.

Proof: For showing that $p : E \to B$ is a fibration consider the diagram

$$\begin{array}{ccccc} W & \overset{s}{\to} & X & \overset{g}{\to} & E \\ & & & f \searrow & \downarrow p \\ & & & & B \end{array}$$

with $s \in S$ and $pgs = fs$. Since $s \in S$, $pgs = fs$ and $p : E \to B$ is a weak fibration, there exist a morphism $g' : X \to E$ and $t : X \to X$ with $t \in S$ such that the following diagram commutes

$$\begin{array}{ccccccc} W & \overset{s}{\to} & X & \overset{t}{\to} & X & \overset{g'}{\dashrightarrow} \atop \underset{g}{\to} & E \\ & & & & & f \searrow & \downarrow p \\ & & & & & & B \end{array}$$

i.e., $g's = gs$, $ts = s$ and $pg' = ft$. It is enough to prove that $pg' = f$. Since $pg' = ft$ we have $pg's = fts = fs$. Since F is a covariant functor, we have $F(pg's) = F(fs)$, i.e.,

$F(p)F(g')F(s) = F(f)F(s)$. Since $F(s)$ is an isomorphism in $\mathcal{C}[S^{-1}]$ we have $F(p)F(g') = F(f)$, i.e., $F(pg') = F(f)$. By Theorem 1.8, F is faithful. Hence we have $pg' = f$. This completes the proof of the Proposition 2.4. ■

3. *S*-cofibrations

The dual concepts of S-fibration and weak S-fibration are respectfully S-cofibration and weak S-cofibration. We recall these concepts from [2].

3.1 Definition. [2] Let S be an arbitrary set of morphisms in a category $\mathcal{C}$. A morphism $j : B \to E \in \mathcal{C}$ is called an S-*cofibration* if for each diagram

$$\begin{array}{ccccc} E & \overset{g}{\rightarrow} & X & \overset{s}{\rightarrow} & W \\ j\uparrow & \nearrow f & & & \\ B & & & & \end{array}$$

with $s \in S$ and $sgj = sf$ there exists a morphism $g' : E \to X$

$$\begin{array}{ccccc} E & \begin{array}{c} g' \\ \dashrightarrow \\ \rightarrow \\ g \end{array} & X & \overset{s}{\rightarrow} & W \\ j\uparrow & \nearrow f & & & \\ B & & & & \end{array}$$

in $\mathcal{C}$ such that $g'j = f$ and $sg = sg'$.

3.2 Definition. [2] A morphism $j : B \to E \in \mathcal{C}$ is called a *weak* S-*cofibration* if for each diagram

$$\begin{array}{ccccc} E & \overset{g}{\rightarrow} & & X & \overset{s}{\rightarrow} & W \\ j\uparrow & & \nearrow f & & & \\ B & & & & & \end{array}$$

with $s \in S$ and $sgj = sf$ there exists a morphism $g' : E \rightarrow X$ and $t : X \rightarrow X$ with $t \in S$

$$\begin{array}{ccccccc} & g' & & & & & \\ E & \overset{\dashrightarrow}{\rightarrow} & & X & \overset{t}{\rightarrow} & X & \overset{s}{\rightarrow} & W \\ & g & & & & & \\ j\uparrow & & \nearrow f & & & & \\ B & & & & & & \end{array}$$

such that $st = s$, $g'j = tf$ and $sg = sg'$.

The following result is elementary in nature.

3.3 Proposition. *S-cofibration implies weak S-cofibration.*

Proof. Let $j : B \rightarrow E$ be an S-cofibration in the category $\mathcal{C}$. In order to show that $j : B \rightarrow E$ is also a weak S-cofibration consider an arbitrary diagram

$$\begin{array}{ccccc} E & \overset{g}{\rightarrow} & & X & \overset{s}{\rightarrow} & W \\ j\uparrow & & \nearrow f & & & \\ B & & & & & \end{array}$$

with $s \in S$ and $sgj = sf$. Since $j : B \rightarrow E$ is an S-cofibration, there exists a morphism $g' : E \rightarrow X$

$$\begin{array}{ccccc} E & \overset{g'}{\dashrightarrow} \atop \underset{g}{\rightarrow} & X & \overset{s}{\rightarrow} & W \\ j\uparrow & \nearrow f & & & \\ B & & & & \end{array}$$

in $\mathcal{C}$ such that $g'j = f$ and $sg = sg'$. Considering $t = 1_X : X \to X$, we can have $st = s$, $g'j = tf$ and $sg = sg'$. ■

Under some moderate assumptions on the set S, it can be proved that weak S-cofibration always implies S-cofibration.

3.4 Proposition. *Let* S *be the set of morphisms in* $\mathcal{C}$. *Let* $F_S = F : \mathcal{C} \to \mathcal{C}[S^{-1}]$ *be the canonical functor. Suppose the following conditions hold*:

(a) $j : B \to E$ *is a weak* S*-cofibration*.

(b) S *admits a calculus of left fractions*.

(c) S *consists of monomorphisms.*

Then $j : B \to E$ *is an* S*-cofibration*.

Proof. For showing that $j : B \to E$ is an S-cofibration, consider an arbitrary diagram

$$\begin{array}{ccccc} E & \overset{g}{\rightarrow} & X & \overset{s}{\rightarrow} & W \\ j\uparrow & \nearrow f & & & \\ B & & & & \end{array}$$

with $s \in S$ and $sgj = sf$. Since $s \in S$ and $sgj = sf$ and $j : B \to E$ is a weak S-cofibration, there exist a morphism $g' : E \to X$ and $t : X \to X$ with $t \in S$ such that the following diagram commotes

$$\begin{array}{ccccccc} & g' & & & & & \\ E & \overset{\dashrightarrow}{\rightarrow} & X & \overset{t}{\rightarrow} & X & \overset{s}{\rightarrow} & W \\ & g & & & & & \\ j\uparrow & & \nearrow f & & & & \\ B & & & & & & \end{array}$$

i.e., $st = s$, $g'j = tf$ and $sg = sg'$. It is enough to prove that $g'j = f$. Since $g'j = tf$ we have $sg'j = stf = sf$. Since F is a covariant functor we have $F(sg'j) = F(sf)$, i.e., $F(s)F(g')F(j) = F(s)F(f)$. Since $F(s)$ is an isomorphism in in $\mathcal{C}[S^{-1}]$ we have $F(g')F(j) = F(f)$, i.e., $F(g'j) = F(f)$. By Theorem 1.8, F is faithful. Hence we have $g'j = f$. This completes the proof of the Proposition 3.4. ■

4. Adams completion and S-fibrations

In [2], Bauer and Dugundji have examined the notion of S-fibration in the category $\mathcal{T}$, the category of topological spaces and continuous functions; under suitable choice of the set S they have shown that a map $p : E \to B$ is an S-fibration if and only if it is a Hurewicz fibration. In this note, under reasonable assumptions we show that a morphism $p : E \to B$ in a category $\mathcal{C}$ is an S-fibration if and only if it is a weak S-fibration.

4.1 Theorem. *Let S be a saturated family of morhpisms of a category $\mathcal{C}$ and let every object in $\mathcal{C}$ admit an Adams completion. Let S consist of monomorphisms. Then $\{$weak S-fibrations$\} = \{S$-fibrations$\}$.*

Proof. The proof follows from Theorem 1.7, Propositions 2.3 and 2.4. ■

The following is a direct consequence of Theorem 4.1.

4.2 Corollary. *Let $\bar{S}$ be the saturation of a family of morhpisms of a category $\mathcal{C}$ and let every object in $\mathcal{C}$ admit an $\bar{S}$-completion. Let S consist of monomorphisms. Then {weak $\bar{S}$-fibrations} = {$\bar{S}$-fibrations}.*

4.3 Note. In the presence of the conditions of Proposition 2.4, we have {weak S-fibrations} = {S-fibrations}.

4.4 Note. If S contains only the identities of the category $\mathcal{C}$, then {weak S-fibrations} = {S-fibrations} ([2], Remark 1); this is so because S satisfies the conditions of Propositions 2.4.

4.5 Remark. Everything which has been obtained for S-fibration and weak S-fibration can be dualized in the usual fashion to yield the corresponding results for S-cofibration and weak S-cofibration [2].

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