A New Set of Codes with Swift Decoding for Overloaded Synchronous CDMA

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***Abstract*—We consider the designing of a new set of uniquely decodable codes for uncoded synchronous overloaded CDMA system that exists for arbitrary values of spreading gain (code length). A fast and recursive method of construction is proposed where the orthogonal Hadamard matrix of least dimension (two), as the basis of construction regularizes a ternary pattern which is further leveraged to attain a rich simplicity in decoder design. The simplicity gained in designing of the proposed Comparison Aided Decoder (CAD) is prominent enough to neglect the marginal sacrifice in Bit Error Rate (BER) as compared to the optimum Maximum Likelihood Decoder (MLD) for noisy transmission. Despite its low complex nature, the detection retains it uniquely decodable (errorless) attribute in the absence of noise. Moreover, for large dimension of the proposed matrices, the loading capacity of the system as compared to the conventional CDMA gets a two fold enhancement.**

I. Introduction

The lack of coexistence of the overloaded dimension, and orthogonality within a single code space drives the motivation towards the involvement of Uniquely Decodable Codes (UDC) sets (matrices) in Code Division Multiple Access (CDMA), where the overloaded dimension of a code matrix refers to the number of codes (signatures) being greater than the spreading gain or code length or total number elements (chips) in a code sequence. A matrix $\mathbf{C}$ is considered as uniquely decodable over X, if for $X_1 \neq X_2$ the inequality $\mathbf{C}X_1 \neq \mathbf{C}X_2$ is true, where X_1 and X_2 denote two different input vectors. In other words, a Uniquely Decodable (UD) matrix is injective in nature or there exists one-to-one mapping between the input and output.

The fact that boosts the gravity of the UD matrices is the domain of its application that not only gets limited to the block coding for T-user multiple access channel [1]–[5] but also have homogeneous employment in CDMA [6]–[12]. Particularly, the use of ternary UD matrices [1]–[6] in the context of multi-user coding has drawn significant attention even if they are of less importance in the coin-weighing problem [13]. For recursive construction based ternary [1]–[3], [6] and binary [9], [11] matrices $\mathbf{C}^k_{N_k \times M_k}$, the overloading factor (total sum rate) denoted as $\beta = (M_k/N_k)$ is shown to be asymptotically equal to that of the maximal achievable sum rate $S_{sum}(k)$, as the value of M_k increases i.e.;

$$\beta \sim 1/2\,(\log_2 M_k) \quad \text{as} \quad M_k \to \infty$$

where
$$S_{sum}(k) = \sum_{f=0}^{M_k} \frac{\binom{M_k}{f}}{2^{M_k}} \log_2 \frac{2^{M_k}}{\binom{M_k}{f}}.$$

Although such constructions theoretically facilitate a large capacity, their decoding for noisy transmission has always been a greater challenge to deal with. For noisy channel, the proposed decoders stand ineffective to provide an acceptable error performance. In general, the factor that implicitly affects the efficiency of the whole system is the decoder, which must be simple in design and perform intrinsically better for noisy environment. Therefore, for such matrices, use of optimum Maximum Likelihood Decoder (MLD) [14] is obviously the only solicited option. However, the sheer impracticality associated with its implementation due to the catastrophic rise in complexity over the linear decoders makes it imperative to look for its suitable substitutes. Ironically, the design of such detectors to meet an acceptable level of Bit Error Rate (BER) in conjunction with the dramatic reduction in complexity is usually challenging and has drawn the attention of many researchers in the past [15], [16].

More recently, new binary, and ternary UD constructions have been proposed for CDMA systems, where the focus is on designing of feasible decoders for better error performance at the cost of sacrificing the asymptotic equality between β and $S_{sum}(k)$. In [7], [10], [17], a two-stage Simplified MLD (SMLD) is proposed for tensor product based matrix construction where additional users are usually kept to a suitable minimal value in order to have better error performance. This is because for noisy channel, the effect of the Multiple Access Interference (MAI) becomes more prominent, even if its impact remains silent in the absence of noise. Likewise, in [6], [18], a highly simplified Logical Decoder (LD) [18] for a class of ternary code sets relies on the Analog to Digital Converter (ADC) to alleviate the effect of noise. For the use of ADC in retaining the constellation of noisy received signal close to that of transmission, the range of constellation values of the transmitted vector must be finite and predictable. For its embodiment in decoding section, all that required is the availability of a regularized and advantageous geometric pattern among the elements (non-zero and zero) of the matrix.

In this paper, we propose a new set of UDC, whose method of construction is fast and recursive. Unlike the UDC sets in [1], [2], [5]–[7], [9], [11], [12], the proposed design being fully governed by a basis set follows a column overlapping

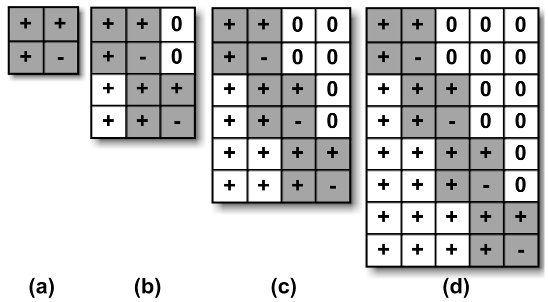


Fig. 1: Proposed matrices for $\mathbf{H}_2$ as the basis set (a) $(\mathbf{C}^1)^T$ (β=1) (b) $(\mathbf{C}^2)^T$ (β=1.33) (c) $(\mathbf{C}^3)^T$ (β=1.5) (d) $(\mathbf{C}^4)^T$ (β=1.6)

mechanism as shown in Fig. 1. Orthogonal Hadamard matrix of smallest dimension i.e., $\mathbf{H}_2$ is considered as the basis. Such a construction hierarchy interestingly approves its existence for arbitrary values of spreading gain and thus expanding its scope towards optimizing the rate-capacity trade off[1]. In deed, it is the ternary pattern evolved in code space followed by the unique encoding capacity of the Hadamard matrix that prompts the decoder to achieve the errorless recovery in the absence of noise. For noisy channel, where the systems in [1]–[5], [11], [12] and [7], [8], [10], [17] relies on the optimum MLD and SMLD respectively, use of comparison driven based logic for decoding dramatically reduces the overall complexity.

II. Signature Matrix Design

The synchronous CDMA system using the proposed code matrix with index-k (i.e. $\mathbf{C}^k$ or $\mathbf{C}_{N_k \times M_k}$) can be modeled as

$$Y = R_k + n \tag{1}$$

where $R_k = \mathbf{AC}X$ is the noiseless received vector for $\mathbf{A} = \mathbf{I}_{N_k \times N_k}$ = Identity Matrix with diagonal elements representing the amplitudes assuming the system to be perfectly power controlled. $X_{M_k \times 1} \in \{\pm 1, 0\}^{M_k}$ is the input column vector, and n denotes the power spectral density of AWGN channel. Following relation describes the recursive logic for construction of $\mathbf{C}^k$ from $\mathbf{C}^{k-1}$

$$\mathbf{C}^k_{N_k \times M_k} = \begin{vmatrix} \mathbf{C}^{k-1}_{(N_k-1)\times(M_k-2)} & \mathbf{1}_{(N_k-1)\times 2} \\ 0_{1\times(M_k-2)} & [\ 1 \quad -1\ ] \end{vmatrix} \text{for} \quad k \in Z^+ \tag{2}$$

where $\mathbf{C}^1 = \mathbf{H}_2$ = basis matrix for construction (Fig. 1 (a)) and M_k, N_k as a function of the matrix index (k) can be defined as

$$N_k = k + 1, \ \ M_k = 2k \ .$$

This is easy to verify that with increase in value of k the rise in the value of β is observed, such that with k approaching to infinity and value of $\beta = \left(\frac{2k}{k+1}\right)$ tends to two. For simplicity in further analysis, all signatures in $\mathbf{C}^k$ can be classified into k different classes for $N_1 < N_2 < \cdots < N_k$, such that

$$\mathbf{C}^k = [\mathbf{C}_1|\mathbf{C}_2|...|\mathbf{C}_k] \tag{3}$$

[1]rate-capacity trade off refers to the proportional reduction in transmission rate incurred due to the selection of code matrix with higher spreading gain (N_k) for enhanced user capacity (M_k).

where $\mathbf{C}_a$ for $a = 1, 2, ..., k$ represents the class with effective spreading gain[2] $N_{ef} = N_a$. The uniqueness in construction also lies in the subset based relationship between the matrices of consecutive index that can be alternately defined by the following recursive logic.

$$\left[\left[\mathbf{C}^{k-1} 0_{1\times(M_k-2)}\right] | \mathbf{C}_k\right] = \mathbf{C}^k \tag{4}$$

The proposed code set $\mathbf{C}^k$ in (2) is uniquely decodable over a set of m input symbols $\psi = \{\xi_1, \xi_2, \cdots \xi_m\}$ such that $\psi \subset \{\alpha_1, \alpha_2, \cdots, \alpha_n, \bar{\alpha}\}$ for $\bar{\alpha}$ being the linear combination over the set of Algebraically Independent Numbers[3] (AIN) $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. In the next section, assuming input system to be binary ($\psi = (1, -1)$), we show the proposed decoding to errorless for noiseless application.

III. Comparison Aided Decoder (CAD)

In this section, design of the proposed decoder is explained where the system is assumed to be fully loaded i.e. $X \in \{\pm 1\}^{M_k}$. Prior to that, we study the pattern of MAI. This is because the quality of recovery of an user in a CDMA system is mostly influenced by the extent of MAI, for which the level of net cross-correlation is usually considered as a metric. In the proposed construction (Fig. 1), we aim to realize this pattern straight from the expression of the recurrent construction explained by (4).

In (4), the zero vector $0_{1\times(M_k-2)}$ adjoined to $\mathbf{C}_{k-1}$ for construction of $\mathbf{C}^k$ splits R_k into two unequal sections. The first N_{k-1} chips carry the data of the M_{k-1} users common to $\mathbf{C}^k$ and $\mathbf{C}^{k-1}$ whereas the last chip have the data of the users of class $\mathbf{C}_k$ only. Equivalently, a section of the spread data of the class $\mathbf{C}_k$ exists in the last most and first $N_{k-1} = (N_k - 1)$ chips of R_k with zero MAI and MAI from $\mathbf{C}^{k-1}$ respectively. Recursively, similar logic also holds for the lower indexed matrices $\mathbf{C}^a$ for $1 < a \leq (k-1)$.

Please note the difference between $\mathbf{C}^a$ and $\mathbf{C}_a$ i.e. $\mathbf{C}_a$ is a subset of the matrix $\mathbf{C}^a$ e.g.; $\mathbf{C}^a = [\mathbf{C}_1|\mathbf{C}_2|...|\mathbf{C}_a]$

Before defining the decoding algorithm for noisy transmission in Table I, we present the following theorem to prove errorless behavior of the decoder in absence of noise. First, we propose Lemma 1 that will supplement the proof in Theorem 1.

Lemma 1: The first non-zero element encountered during the traversal from chip-N_k to chip-1 of the transmitted total sum vector R_k suffices for the errorfree decoding of the last class in $\mathbf{C}^k$ i.e., $\mathbf{C}_k$.

For the proof, please refer to Appendix A

Theorem 1: For the proposed UDC matrix $\mathbf{C}^k$, the decoded input vector $\hat{X} \in \{\pm 1\}^{M_k}$ from R_k is errorfree.

[2]Effective Spreading Gain (N_{ef}) of a signature (ternary) in $\mathbf{C}^k$ indicates the total number of chips occupied by the non-zero elements (i.e. 1 or -1).

[3]For an algebraically independent set, the linear combinations of the numbers with integer coefficients become zero.

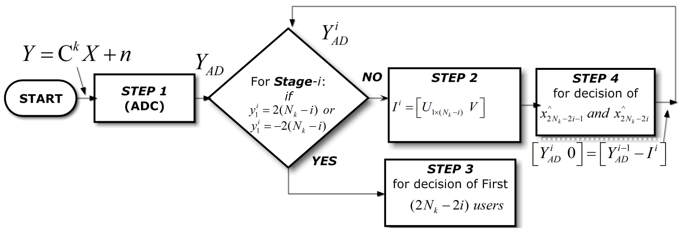


Fig. 2: Sequential Flow Diagram for CAD (Table I)

Proof : For proof, we carry an induction on k. From (1) and (3), the noiseless vector R_k is the summed contribution of the received vectors from k-different classes of $\mathbf{C}^k$ e.g.,

$$R_k = \sum_{a=1}^{k} R_{N_a} \text{ for } R_{N_a} = \mathbf{A}_a \mathbf{C}_a X_a \tag{5}$$

where $\mathbf{A}_a = \mathbf{I}_{h\times h}$, $X_a \in X = [X_1 X_2 \cdots X_k]^T$. In Lemma 1, we have shown that the last class of the proposed matrix $\mathbf{C}^k$ i.e., $\mathbf{C}_k$ can be extracted from R_k without any error ($\hat{X}_k = X_k$). In fact, this itself is the first stage of decoding ($i = 1$). Next, estimation of its interference on other existing classes is evaluated as $I = \mathbf{C}_k \hat{X}_k$. The estimated interference I is then subtracted from R_k to produce R_{k-1} such that

$$[R_{k-1}\ 0] = [R_k - I]\,.$$

Conceptually, following the recursive structure in (4), R_{k-1} denotes the received vector for $\mathbf{C}^{k-1} = \left[\mathbf{C}^1|\mathbf{C}^2|\cdots|\mathbf{C}^{k-1}\right]$ e.g., $R_{k-1} = \mathbf{A}_{k-1}\mathbf{C}^{k-1}X_{(M_k-2)\times 1}$. Since the last class of $\mathbf{C}^{k-1}$ is $\mathbf{C}_{k-1}$, the newly generated vector R_{k-1} if exposed to the proposition of Lemma-1 will result in the errorless decoding of X_{k-1}. This becomes the second stage of decoding ($i = 2$). Without loss in generality, this sequence of Detection (of a class)-Estimation (of its MAI on other classes)-Cancellation (of the estimated MAI) is to be continued until the class $\mathbf{C}_1$ gets retrieved from the respective decoding vector $R_1 = \mathbf{A}_1\mathbf{C}^1 X_1$ in k^{th} stage of decoding ($i = k$). This completes the proof. For better organization of the analysis, i is used to indicate the decoding stage and also termed as *decoding index*. ■

From Theorem 1, we derive the steps corresponding to decoding of the noisy vector Y and present it in Table I. Figure 2 shows the respective flow diagram.

IV. Computational Complexity

From step-2 of CAD, the prevailing pattern among the signatures of each class can be exploited to make the estimation of I^i void of multiplications. Therefore, the overall complexity gets limited to comparisons and additions only. Here, we consider the noiseless case for estimation of the complexity. For noisy scenario, the variation is highly marginal depending on the error introduced at ADC (in step-1) for $Y_{AD} \neq R_k$.

For calculation of the total number of comparisons (P_X), the input vector X can be split based on the decision making

TABLE I

CAD Algorithm

Step 1: Allow Y in (1) to pass through a $(M_k + 1)$-ary ADC to deliver
$$Y_{AD} = \left[y_1 y_2 \cdots y_{N_k}\right]$$
where $y_j \in \{0, \pm 2, \pm 4, \cdots, \pm(2N_k - 2j + 2)\}$ for $1 < j \leq N_k$
and
$$Range\,(y_1) = Range\,(y_2)$$

Step 2: For stage-i $(1 < i \leq k)$, Find

- $I^i = \mathbf{C}_{k-i+2}\hat{X}_{k-i+2} = \left[U_{1\times(N_k-i)}\ V\right]$

  where $\left(\hat{x}_{2N_k-2i-1}, \hat{x}_{2N_k-2i}\right\} \in \hat{X}_{k-i+2}$
  $$U = \left(\hat{x}_{2N_k-2i-1} + \hat{x}_{2N_k-2i}\right),$$
  and
  $$V = \left(\hat{x}_{2N_k-2i-1} - \hat{x}_{2N_k-2i}\right)$$

- $Y_{AD}^i = \left[y_1^i y_2^i \cdots y_{N_{k-i+1}}^i\right]$ e.g.,
  $$\left[Y_{AD}^i 0_{(h-1)\times 1}\right] = \left[Y_{AD}^{i-1} - I^i\right]$$
  (For $i = 1$, $Y_{AD}^1 = Y_{AD}$, $I_1 = 0_{N_k\times 1}$)

Step 3: For Y_{AD}^i
If $y_1^i = 2(N_k - i)$
then $\left[\hat{x}_1, \hat{x}_2 \cdots \hat{x}_{2N_k-2i-1}, \hat{x}_{2N_k-2i}\right] = [1]_{2N_k-2i}$,
Else If $y_1^i = -2(N_k - i)$
then $\left[\hat{x}_1, \hat{x}_2 \cdots \hat{x}_{2N_k-2i-1}, \hat{x}_{2N_k-2i}\right] = [-1]_{2N_k-2i}$
Thus, X is completely decoded
and no need of any further stages. Otherwise, follow *Step 4*.

Step 4: For Y_{AD}^i, traverse from its $chip - (N_k - i + 1)$ to $chip - 1$ and decipher $chip - p$, such that $Y_{AD}^i(p) \neq 0$ and verify the following.

If $Y_{AD}^i(p) = 2\ or\ 4$
then $\left(\hat{x}_{2N_k-2i-1}, \hat{x}_{2N_k-2i}\right) = (1, 1)$

Else if $Y_{AD}^i(p) = -2\ or\ -4$
then $\left(\hat{x}_{2N_k-2i-1}, \hat{x}_{2N_k-2i}\right) = (-1, -1)$

Go to *Step-2* and repeat the sequence of steps for stage-$(i + 1)$.

Finally, $\hat{X} = \left[\hat{X}_1 \hat{X}_2 \cdots \hat{X}_k\right]^T$ is the decoded input vector

steps i.e.; $X = \{X_{step-3}|X_{step-4}\}$. In other way, $k = p + q$, where p and q indicate the number of classes included in X_{step-3} and X_{step-4} respectively. Thus,

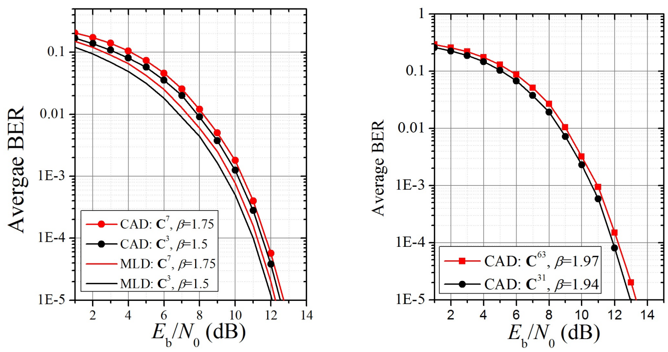
$$P_X = P_{X(step-3)} + P_{X(step-4)}. \tag{6}$$

In (6), $P_{X(step-3)} = 1$, if and only if X contains a series of 1 or -1 for consecutive classes starting from X_1 i.e., $\{X_1\} = 1_{2\times 1}$or $-\ 1_{2\times 1}$, $\{X_1, X_2\} = 1_{4\times 1}$or $-\ 1_{4\times 1}$, $\{X_1, X_2, X_3\} = 1_{6\times 1}or-1_{6\times 1}$, …, $\{X_1, X_2, X_3, \ldots, X_k\} = 1_{M_k\times 1}$or $-\ 1_{M_k\times 1}$ for which $p = 1, 2, 3, \ldots, k$ respectively. Otherwise, $P_{X(step-3)} = 0$. Unlikely, the value of $P_{X(step-4)}$ is the sum of all the comparisons needed for each stage individually, where step-4 is required for decoding. So, inclusion of more classes to X_{step-3} in turn minimizes the number of calls made to step-4 (i.e., q) and thus simplifies the overall decoding. Interestingly, for $X_{step-3} = X$ i.e., $P_X = 1$, $Q_X = 0$ which implies that only a comparison decides for the input vector $\hat{X}$.

Similar to P_X, total number of additions (Q_X) is also input combination variant. However, the noteworthy point is that addition operation is demanded only for the decoding of the q consecutive classes of X_{step-4}. As evident from Table

TABLE II: CAD versus MLD: Complexity Analysis

Decoder	Size	Multiplications	Additions (Q_X)	Comparisons (P_X)
CAD	4x6	Nil	12.19	6.75
	8x14	Nil	27.58	21.64
MLD	4x6	184	201	64
	8x14	104686	124690	16384



(a) (b)

Fig. 3: BER versus (E_b/N_0) performance (a) of the Proposed Matrix (CAD and MLD) for $\mathbf{C}^3$, $\mathbf{C}^7$ with size (4×6), (8×14) (b) of CAD for $\mathbf{C}^{31}$, $\mathbf{C}^{63}$ with size (32×62), (64×126)

I, decoding in such case is achieved by the joint contribution of step-2 and step-4. Where step-2 is meant for estimation and cancellation of the interference, step-4 process the outcome of step-2 to offer the final decision. Thus, we can express Q_X as

$$Q_X = Q_{IE} + Q_{IC} \tag{7}$$

where Q_{IE}= number of additions for Interference Estimation (IE) = $2q$. $Q_{IC} = \sum_{e=1}^{q} N_{k-e+1}$= number of additions involved in q stages of Interference Cancellation (IC). For better perception of the overall simplicity of CAD, a comparative study of complexity with respect to optimum MLD is presented in Table II.

From the above analysis, we realize that it is the simplicity of construction, and pattern of the orthogonal Hadamard set ($\mathbf{H}_2$), which leads to the swiftness of the decoder, as shown in Table II.

V. Simulation Results

In this section, the BER vs (E_b/N_0) performance for AWGN channel is discussed. The system is assumed to be BPSK modulated and perfectly power controlled. Table II presents the overall complexity, where P_X in (6) and Q_X in (7) are calculated for each of the 2^{M_k} combinations of X and averaged.

From Fig. 3 (a), the error performance of CAD shows a lag of ≈ 0.5 dB in signal to noise ratio (SNR) at a BER of 10^{-4}. The loss incurred, if compared with the extensive lowering in complexity (in Table II), is worthy to be overlooked. Moreover, unlike optimum MLD, no need of multiplication is demanded for CAD. Along with, the degradation in average BER with respect to the increase in value of β (from 1.5 to 1.75) reported by the curves is highly marginal .

In Fig. 3 (b), the BER performance of CAD for high dimensional matrices $\mathbf{C}^{31}$ and $\mathbf{C}^{62}$ sized as (32×62) and (64×126) ($\beta = 1.94$ and 1.97) are shown. Although the level of BER subjected to the significant increase in value of β increases, the rise as compared to that of the lower dimension is found to be negligible. Furthermore, the use of ADC at the decoder (Fig. 2) aims at minimizing the error ($Y_{AD} \neq R_k$). At high value of (E_b/N_0), the decoder prohibits the BER performance from deviating significantly. This is because at this level of (E_b/N_0), the constellation of the noisy received signal becomes close to that of the noise less case i.e., $Y_{AD} \approx R_k$.

VI. Conclusion

In this paper, we have shown that, there prevails a particular set of ternary uniquely decodable codes for overloaded synchronous CDMA that substantiates its existence for arbitrary values of rate factor. Additionally, the uniqueness of employing a binary (Hadamard) matrix as the basis not only made the overall construction simplified but also empowered it to double its capacity i.e. $\beta \approx 2$. Due to the comparison driven logic, the swiftness of CAD improved dramatically. We also proved the errorfree nature of recovery for noiseless transmission. Despite having the significant simplicity (Table II), the deviation in error performance as compared to the optimum MLD is highly marginal. Simulation results also verify the noteworthy competence of its BER performance for large dimensional code sets.

Appendix A

According to (5), the following expressions can be explicitly inferred for $R_{N_a} \in \{\pm 2, 0\}^{N_a}$ corresponding to class $\mathbf{C}_a$.

$$R_{N_a}(N_a) = X_a(1) - X_a(2) \tag{8}$$

$$R_{N_a}(N_a) = X_a(1) + X_a(2) \tag{9}$$

where $1 \leq a \leq k$ and $X_a(b)$ for b = 1, 2 represents the two users of $\mathbf{C}_a$. Recall that N_a denotes the length of the non-zero sequence of the signatures in $\mathbf{C}_a$. Hence, $R_{N_a}(N_a)$ refers to the last non-zero chip of the sum vector for $\mathbf{C}_a$. For remaining $(N_a - 1)$ number of chips with non-zero signal levels, the constellation pattern can be predicted as

$$\begin{array}{ll} R_{N_a}\left(N_a\right) = 0 & \Rightarrow \; R_{N_a}\left(c\right) = 2 \text{ or} -2 \\ R_{N_a}\left(N_a\right) = 2 \text{ or} -2 & \Rightarrow \; R_{N_a}\left(c\right) = 0 \end{array} \tag{10}$$

where $c = 1, 2, ..., (N_a - 1)$.

With these preliminaries, now, we intend to trace the variation in constellation values among all the N_k non-zero chips of the transmitted total sum vector R_k. Note that our focus is to correctly decode the last class of the whole matrix only i.e., $\mathbf{C}_k$. Now, we deduce the following relation in (11) to define the constellation value at chip-$(N_k - s)$ of R_k, which is a function of the sum vectors of participating classes such that

$$R_k(N_k - s) = \sum_{t=0}^{s} R_{(N_k - t)}\left(N_k - s\right) \tag{11}$$

Onwards, the proof to show $\hat{X}_k = X_k$ is based on induction on s. So, we present Table III as an expanded

TABLE III: Study of constellation pattern of the total sum vector $R_k(N_k - s)$ in (11), as a function of the sum vector of the individual constituent classes, subjected to variation in value of s ($0 \leq s \leq 3$)

s	$R_{N_k}(N_k - s)$	$R_{N_{k-1}}(N_k - s)$	$R_{N_{k-2}}(N_k - s)$	$R_{N_{k-3}}(N_k - s)$	$R_k(N_k - s)$	Status of Decoding
0	2 / -2	-	-	-	2 / -2	Y
	0	-	-	-	0	N
1	2	0 / 2	-	-	2 / 4	Y
	-2	0 / -2	-	-	-2 / -4	Y
	2	-2	-	-	0	N
	-2	2	-	-	0	N
2	2	0	0 / 2	-	2 / 4	Y
	-2	0	0 / -2	-	-2 / -4	Y
	2	0	-2	-	0	N
	-2	0	2	-	0	N
3	2	0	0	0 / 2	2 / 4	Y
	-2	0	0	0 / -2	-2 / -4	Y
	2	0	0	-2	0	N
	-2	0	0	2	0	N

overview of the expression in (11) with an intention to study the pattern of constellation of R_k subjected to all possible constellation values of the sum vectors of constituent classes, for specific values of s. Following an uniform approach of analysis, we start with s = 0 and proceed till $s = k$. However, the information in Table III covers for the first four values of s ($0 \leq s \leq 3$) in a top to bottom chronology. This is because, our objective is just to study the pattern of the constellation for different values of s. To accomplish this consideration of the first four value of s are sufficient, since similar behavior in the pattern can be expected for higher values of s too.

The rows corresponding to a particular value of s carry all possible combinations of the sum vector from $(s+1)$ classes contributing to the level of $R_k(N_k - s)$. The contents of the last (rightmost) column indicates the status of decoding: unambiguous or ambiguous, denoted by "Y" or "N" respectively. The symbol (-) has been used to mark the specific classes, not contributing to the final sum $R_k(N_k - s)$.

As evident, with the increase in value of s the number of classes or class wise sum vectors leading to R_k increases. Following a possible approach, the decoding analysis for "Y" or "N" can be switched to a next value of s, if and only if there appears an ambiguity ("N" in Table III) with respect to its present value. In the following, we explain the proposed analysis to be made for each value of s. An elaborated overview of the whole analysis with respect to s is discussed below.

For $s = 0$, if $R_k(N_k) = R_{N_k}(N_k)$ = 2 and -2 then $(X_k(1), X_{N_k}(2)) = (1, -1)$ and $(-1, 1)$. Thus, decoding is errorless (unambiguous), as shown by "Y" in first row of for $s = 0$. In contrast, For $s = 0$, if $R_k(N_k) = 0$ then there exists an ambiguity for $(X_k(1), X_k(2)) = (1, 1)$ or $(-1, -1)$, shown by "N" in second row for $s = 0$. To resolve this, s is to be incremented by 1, such that for $s = 1$ and $R_k(N_k) = 0$

$$R_k(N_k - 1) = R_{N_k}(N_k - 1) + R_{N_k - 1}(N_k - 1) \qquad (12)$$

According to (10), now, value of $R_{N_k}(N_k - 1) = 2 or - 2$, since $R_k(N_k) = 0$. Under such conditions, if $(R_{N_k}(N_k - 1), R_{N_k - 1}(N_k - 1))$ becomes (2, 0), (2, 2), (-2, 0), (-2, -2) further leading to $R(N_k - 1)$= 2, 4, -2, -4, then $\left(\hat{X}_{N_k}(1), \hat{X}_{N_k}(2)\right)$ = (1,1), (1,1), (-1,-1) and (-1,-1) respectively, shown by "Y", where as for $(R_{N_k}(N_k - 1), R_{N_k - 1}(N_k - 1))$ = (2,-2) and (-2,2), $R(N_k - 1) = 0$ asserts an ambiguity, reported by "N" (see Table III, $s = 1$). To overcome this, s is again incremented by 1, such that for $s = 2$ and $R_k(N_k) = R_k(N_k - 1) = 0$

$$R_k(N_k - 2) = R_{N_k}(N_k - 2) + R_{N_k - 1}(N_k - 2) + R_{N_k - 2}(N_k - 2) \qquad (13)$$

Following (10), since $R_{N_{k-1}}(N_k - 1) = 2$ or -2 implies $R_{N_{k-1}}(N_k - 2) = 0$ (also see Table III, $s = 2$), the expression in (13) becomes

$$R_k(N_k - 2) = R_{N_k}(N_k - 2) + R_{N_k - 2}(N_k - 2) \qquad (14)$$

The resulting expression in (14) has complete similarity to that of (12). Hence it is trivial to expect the analysis for decoding from (14) to be identical to that of (12).

From Table III, we note that for higher values of $s(> 1)$, the analysis aiming for the unambiguous decoding although involves more number of classes (> 2), the final sum element of R_k carries a value equal to the addition of sum vectors of two class only, one of which is always the last class $\mathbf{C}_k$, by default. Therefore, for a given transmitted vector R_k, this is feasible to decipher the input vector of the last class ($\hat{X}_k$) following the construction and analysis of equations similar to (12) and (14). Furthermore, this process continues till $s = (N_k - 1)$, unless a non-zero level of constellation is found for $R_k(N_k - s)$ i.e., $\hat{X}_k$ is decoded. In particular, when all last $(N_k - 1)$ chips of R_k fails to offer the unambiguous recovery due to the absence of non-zero elements i.e., for $s = N_k - 2$ and $R_k(N_k) = R_k(N_k - 1) = \ldots = R_k(2) = 0$, there always exists a non-zero element at $R_k(1)$ that leads to the correct decoding. Thus, feasibility of the errorless decoding for the last class in $\mathbf{C}^k$ is verified. ■

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