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Layer-adapted meshes for parameterized singular perturbation problem

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Abstract

A nonlinear singularly perturbed boundary value problem depending on a parameter is considered. Two numerical methods are applied to solve this problem. First, we solve the problem using backward Euler finite difference scheme on layer adapted meshes. Then, Richardson extrapolation technique is applied to improve the accuracy of the computed solution. Numerical experiment is carried out to validate the theoretical estimates.

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Keywords: Parameterized problem; Boundary layer; Richardson extrapolation; Layer adapted mesh; Singular perturbation.

1. Introduction

Consider the following singularly perturbed boundary value problem (BVP) depending on a parameter:

$$\begin{cases} \varepsilon u'(x) + f(x,u,\lambda) = 0, & x \in \Omega = (0,1), \\ u(0) = s_0, \quad u(1) = s_1, \end{cases} \tag{1}$$

where $0 < \varepsilon \ll 1$ is small and known as the singular perturbation parameter, λ known as the control parameter and $s_0,\ s_1$are given constants. Here, $f(x,u,\lambda)$ is assumed to be sufficiently smooth and satisfying

$$\begin{cases} f(x,u,\lambda) \in C^3([0,1]\times R^2), \\ 0 < \alpha \leq \dfrac{\partial f}{\partial u} \leq \alpha^* < \infty \qquad (x,u,\lambda) \in [0,1]\times R^2, \\ 0 < m \leq \left|\dfrac{\partial f}{\partial \lambda}\right| \leq M < \infty \qquad (x,u,\lambda) \in [0,1]\times R^2. \end{cases} \tag{2}$$

With these assumptions, the BVP (1) possesses a unique solution having a boundary layer of width $O(\varepsilon)$ near $x = 0$ (refer [1,9,10]) . The parameter λ has no connection with the eigenvalue of the nonlinear differential equation. Since there are two unknowns, two boundary conditions are given in (1) to determine it exactly.

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Parameterized boundary value problems have been considered for many years. Existence and uniqueness of the solution for BVP (1.1) was first considered by Pomentale[9]. Jankowski and Lakshmikantham [5], Liu and Mcare [6] constructed a monotonic iterative methods to solve this problem. But the above mentioned papers were only concerned with the regular cases. In recent years, many researchers considered the singular perturbation cases for this problem. Amiraliyev and Duru [1] gave a uniform finite difference method on a standard Shishkin mesh for BVP (1.1) and shown that the method is first order convergent up to a logarithmic factor ($O(N^{-1}\ln N)$). Cen [3] considered a hybrid difference scheme that combines upwind scheme on the fine mesh with the midpoint upwind scheme on the coarse mesh. Xie et. al. [16] used boundary layer correction technique for solving the parameterized problem. Turkyilmazoglu [15] constructed a methodology based on the homotopy analysis technique to approximate the analytic solution.

In this work, we have solved the BVP (1) by backward Euler difference scheme on layer adapted meshes. Then, we have successfully applied the Richardson extrapolation technique on the computed solution to enhance the accuracy. Throughout this paper, 'C' denotes a generic positive constant independent of both ε and N that can take different values at different places. Here, we denote $g(x_i) = g_i$ & $\|\gamma\| = \max|\gamma(x)|,\ \gamma \in C([0,1])$.

2. Numerical Schemes

We shall consider two different numerical schemes on arbitrary nonuniform mesh $\Omega^N : 0 < x_0 < x_1 < \ldots < x_N$, and $h_i = x_{i-1} - x_i$.

2.1. The Backward Euler Scheme

The backward Euler scheme for (1) is given as:

$$\begin{cases} T^N U_i^N \equiv \varepsilon \dfrac{U_i^N - U_{i-1}^N}{h_i} + f(x_i, U_i^N, \lambda^N) = 0\,, \quad 1 \le i \le N-1, \\ U_0^N = s_0, \quad U_N^N = s_1, \end{cases} \tag{3}$$

2.2. Post-processing technique

To increase the accuracy of the difference scheme (3), following the idea of Natividad and Stynes [8], we consider a new nonuniform mesh $\Omega^{2N} : 0 < \widetilde{x}_0 < \widetilde{x}_1 < \ldots < \widetilde{x}_{2N}$ and $\widetilde{h}_i = \widetilde{x}_{i-1} - \widetilde{x}_i$, which is obtained by bisecting each interval of the original mesh Ω^N. Now, we define backward Euler scheme on the mesh Ω^{2N} as:

$$\begin{cases} \widetilde{T}^{2N} \widetilde{U}_i^{2N} \equiv \varepsilon \dfrac{\widetilde{U}_i^{2N} - \widetilde{U}_{i-1}^{2N}}{h_i} + f(\widetilde{x}_i, \widetilde{U}_i^{2N}, \lambda^{2N}) = 0\,, \quad 1 \le i \le 2N-1, \\ \widetilde{U}_0^{2N} = s_0, \quad \widetilde{U}_2^{2N} N = s_1. \end{cases} \tag{4}$$

We are interested in the extrapolated solution defined on the mesh Ω^N by $\overline{U}_i^N = 2\widetilde{U}_i^{2N} - U_i^N$, for $i = 1, 2, \ldots, N$; which we expected will improve the accuracy of the approximation and results in a better approximation than obtained by (3) and also it will enhance the order of convergence.

3. Layer-adapted meshes

Numerical methods using standard finite difference schemes on uniform meshes are inadequate for solving SPPs (refer [4,7,11,14]). This issue can be resolved by using layer adapted meshes. These meshes can be divided into two categories: a priori mesh, for which a prior information about the location and the width of the solution is required and a posteriori mesh, for which we do not need such information.

3.1. Shishkin type meshes

Let τ denotes transition point defined by, $\tau = \min\{\frac{1}{2}, \frac{2\varepsilon}{\alpha}\ln N\}$, which divide Ω^N into two subdomains. On $[0,\tau]$, the mesh will be fine and on $[\tau, 1]$ the mesh will be coarse. On $[0,\tau]$, let our mesh be given by piecewise continuously differentiable and monotonic increasing function ϕ such that $\phi(0) = 0$ and $\phi(1/2) = \ln N$ then the mesh point is given by,

$$x_i = \begin{cases} \dfrac{2\varepsilon}{\alpha}\phi(t_i), & \text{for } t_i = \frac{i}{N},\ i = 0, 1, \ldots, N/2, \\ 1 - \left(1 - \dfrac{2\varepsilon}{\alpha}\ln N\right)\dfrac{2(N-i)}{N}, & \text{for } i = N/2+1, \ldots, N. \end{cases} \tag{5}$$

From this definition we see that for standard Shishkin mesh (S-mesh) $h_i = h = \dfrac{2\tau}{N}$, $i = 0, \ldots, N/2$ and $h_i = H = \dfrac{2(1-\tau)}{N}$, $i = N/2+1, \ldots, N$ and $N^{-1} \le H \le 2N^{-1}$. To define Bakhvalov-Shishkin mesh (B-S-mesh), let us consider a new increasing function 'ψ' that is closely related to ϕ and defined by $\phi = -\ln\psi$ which satisfies $\psi(0) = 1$ and $\psi(\frac{1}{2}) = N^{-1}$, then

$$\psi(t) = e^{(-2\ln N)t} \quad (\text{S-mesh}) \tag{6}$$
$$\psi(t) = 1 - 2(1 - N^{-1})t \quad (\text{B-S-mesh}) \tag{7}$$

3.2. Adaptive grid

Adaptive grid is one of the special kind of non-uniform mesh. A commonly used technique to generate adaptive grid is based on equidistribution of an arbitrary non-negative function $M(u(x), x)$ defined on $[0, 1]$. The monitor functions are usually depends on the gradient of the solution. A grid Ω^N is said to be equidistributed if

$$\int_{x_{j-1}}^{x_j} M(u(s), s)ds = \int_{x_j}^{x_{j+1}} M(u(s), s)ds, \quad j = 1, 2, \ldots, N-1, \tag{8}$$

The solution of (8) along with a discretized version of the BVP (1) produces numerical approximation to the solution of the BVP (1).

4. Error estimates

Proposition 1. *Let $\{u(x), \lambda\}$ and $\{U_i^N, \lambda^N\}$ be the exact solution and discrete solution obtained by the backward Euler method on layer adapted meshes (defined in Section 3) respectively. Then, there exists a constant C such that*

$$\|u_i - U_i^N\| \le CN^{-1}\ln N, \quad |\lambda - \lambda^N| \le CN^{-1} \quad (\textit{S-mesh})$$
$$\|u_i - U_i^N\| \le CN^{-1}, \quad |\lambda - \lambda^N| \le CN^{-1} \quad (\textit{B-S-mesh})$$
$$\|u_i - U_i^N\| \le CN^{-1}, \quad |\lambda - \lambda^N| \le CN^{-1} \quad (\textit{adaptive grid})$$

Proof. One can refer [1, Theorem 1] for S-mesh, [2, Theorem 1] for B-S-mesh and [12] for adaptive grid. ∎

Proposition 2. *Let $u(x), \lambda$ and $\{\overline{U}_i,\ \lambda^N\}$ be the exact solution and discrete solution obtained by the Richardson extrapolation technique on layer adapted meshes (defined in Section 3) respectively. Then, there exists a constant C such that*

$$\|u_i - \overline{U}_i\| \le CN^{-2}ln^2N \quad (\textit{S-mesh})$$
$$\|u_i - \overline{U}_i\| \le CN^{-2} \quad (\textit{B-S-mesh})$$
$$\|u_i - \overline{U}_i\| \le CN^{-2} \quad (\textit{adaptive grid}).$$

Proof. The estimates on S-mesh and B-S-mesh one may refer [13] while the inequality for adaptive grid is being carried out, but the numerical result obtained validate the theoretical estimate

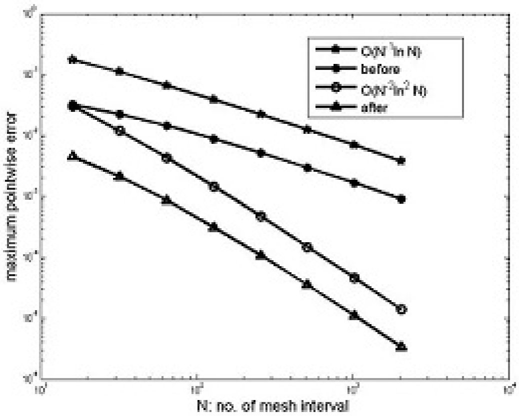

Fig. 1. Comparison of nodal errors on S-mesh.

5. Numerical Results

Here, we consider a test problem to show the applicability and efficiency of the methods on layer adapted meshes.

Example 1. *Consider the following parameterized BVP:*

$$\begin{cases} \varepsilon u'(x) + 2u - \exp(-u) + \lambda = 0, \quad x \in \Omega = (0,1)\,, \\ u(0) = 0, \quad u(1) = 1. \end{cases} \tag{9}$$

Since the exact solution is not available for the BVP (9), we have used interpolation to calculate the error. Define $\widehat{U}_i^{2N}$ piecewise linear interpolation to U_i^N in Ω^N. The maximum pointwise error $E_{\varepsilon,u}^N$ and the rate of convergence $r_{\varepsilon,u}^N$ is defined as:

$$E_{\varepsilon,u}^N = \|U_i^N - \widehat{U}_i^{2N}\|, \quad r_{\varepsilon,u}^N = \log_2\left(\frac{E_{\varepsilon,u}^N}{E_{\varepsilon,u}^{2N}}\right) \tag{10}$$

Table 1 presents the maximum point wise error and corresponding rate of convergence for $\varepsilon = 1e-8$, which is enough to show the singularly perturbed nature of the BVP (9). It shows that after extrapolation nodal error is substantially decreased while the rate of convergence is doubled. Figures 1, 2 and 3 are the graphs of the maximum point-wise errors along with the theoretical rate of convergence on loglog scale for S-mesh, B-S-mesh and adaptive grid respectively. A comparative study shows that the numerical solution before and after extrapolation on the S- mesh leads to larger error than on the B-S-mesh and the adaptive grid. On B-S-mesh and adaptive grid numerical approximation obtain is first order convergence before extrapolation and second order convergence after extrapolation, which are optimal in these cases. But to generate B-S- mesh we need to have a prior information about the location and width of the layer. Advantage of the adaptive grid and B-S mesh over the S-mesh is evident from the theoretical estimates obtained as well as from the numerical results shown. The numerical results are the clear illustration of the error estimates.

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Table 1. $E^N_{\varepsilon,u}$ and the corresponding $r^N_{\varepsilon,u}$ for the Example 1.

Number of intervals N		S-mesh	B-S-mesh	Adaptive grid
	before	3.203e-2	2.849e-2	2.796e-2
16	rate	0.535	0.814	1.069
	after	4.440e-3	3.025e-3	2.776e-3
	rate	1.085	1.668	2.247
	before	2.211e-2	1.621e-2	1.332e-2
32	rate	0.644	0.901	1.106
	after	2.094e-3	9.523e-4	5.849e-4
	rate	1.304	1.820	2.205
	before	1.415e-2	8.679e-3	6.188e-3
64	rate	0.716	0.949	1.029
	after	8.478e-4	2.696e-4	1.269e-4
	rate	1.443	1.909	2.070
	before	8.610e-3	4.497e-3	3.033e-3
128	rate	0.769	0.974	1.012
	after	3.118e-4	7.181e-5	3.023e-5
	rate	1.548	1.954	1.977
	before	5.053e-3	2.289e-3	1.504e-3
256	rate	0.807	0.987	1.003
	after	1.066e-4	1.853e-5e-5	7.678e-6
	rate	1.619	1.977	1.987
	before	2.888e3	1.155e-3	7.506e-4
512	rate	0.835	0.993	1.004
	after	3.474e-5	4.707e-6e-5	1.937e-6
	rate	1.672	1.988	1.969

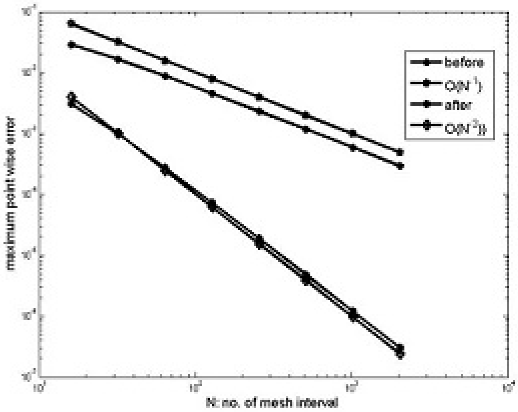

Fig. 2. Comparison of nodal errors on B-S-mesh.

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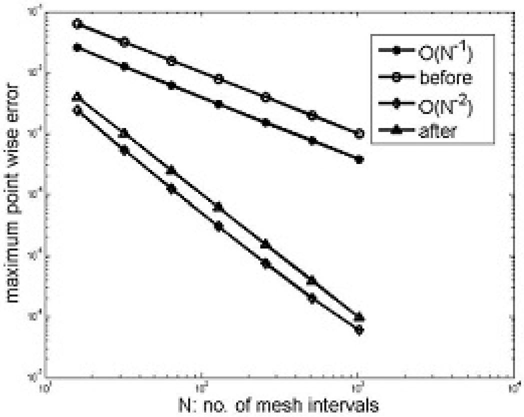

Fig. 3. Comparison of nodal errors on adaptive grid.

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