NUMERICAL SOLUTION OF FRACTIONAL ORDER DIFFERENTIAL-ALGEBRAIC EQUATIONS USING GENERALIZED TRIANGULAR FUNCTION OPERATIONAL MATRICES

S.K. DAMARLA, M. KUNDU

ABSTRACT. This article introduces a new application of piecewise linear orthogonal triangular functions to solve fractional order differential-algebraic equations. The generalized triangular function operational matrices for approximating Riemann-Liouville fractional order integral in the triangular function (TF) domain are derived. Error analysis is carried out to estimate the upper bound of absolute error between the exact Riemann-Liouville fractional order integral and its TF approximation. Using the proposed generalized operational matrices, linear and nonlinear fractional order differential-algebraic equations are solved. The results show that the TF estimate of Riemann-Liouville fractional order integral is accurate and effective.

1. INTRODUCTION

Many physical problems in mechatronics, chemical kinetics, optimal control, electric circuit design, chemical process control, molecular dynamics, incompressible fluids, power systems and industrial production processes etc. are successfully modelled by the use of differential-algebraic equations [1, 2].

Generally the differential-algebraic equations are expressed in a fully implicit form given by

\[ F(t, Y, Y') = 0 \]  \hspace{1cm} (1)

where \( Y \) is a vector of dependent variables, \( F \) is a vector of functions of time \( t \), dependent variables and their derivatives and \( \partial F/\partial Y' \) may be singular. The special case of differential-algebraic equations in [1] which is frequently appearing in practical applications, is semi-explicit differential-algebraic equation or ordinary differential equation with algebraic constraints defined as follows.

\[ Y' = F(t, Y, Z) \]
\[ 0 = g(t, Y, Z) \]  \hspace{1cm} (2)

2010 Mathematics Subject Classification. 26A33, 42C05, 74H15, 35E15.

Key words and phrases. Triangular functions, fractional order differential-algebraic equations, generalized triangular function operational matrices.

The mathematical theory and properties of differential-algebraic equations can be found in Brenan et al. [3], Campbell et al. [4] and Gear [5]. The complexity in solving differential-algebraic equation is usually indicated by differential index. The differential index is the minimum number of differentiations required to convert a given differential-algebraic equation into a system of ordinary differential equations. The differential index of semi-explicit differential-algebraic equation in 2 is 1 if $\frac{\partial g}{\partial Z}$ is non-singular. The high index (index greater than 1) differential-algebraic models are difficult to solve so an alternative is to utilize index reducing method which transforms high index problems into low index (or to one) problems by successive differentiation of algebraic constraints [6].

Several real world phenomena, which exhibit fractional dynamics, in biology, electrochemistry, electromagnetism, acoustics, material science, control theory, physics etc. are well represented by fractional order derivative than the classical one [7, 8, 9, 10, 11]. The non-local nature of fractional order derivative made it capable of explaining the inherent fractional order description of such physical processes. Electrochemical processes, non-integer order optimal controller design, complex biochemical processes may find application of fractional differential algebraic equations.

In this article, we consider the following fractional order differential-algebraic equations.

\[
\begin{align*}
D^{\alpha_i} x_i (t) &= f_i (t, x_1, x_2, \ldots, x_n) \\
0 &= g (t, x_1, x_2, \ldots, x_n)
\end{align*}
\]

(3)

with initial conditions $x_i (0) = a_i, i = 1, 2, \ldots, n$.

Here $D^{\alpha_i}$ is Caputo fractional derivative of order $\alpha_i$ satisfying the relation $m - 1 < \alpha_i \leq m, m \in \mathbb{N}$.

In general, most fractional order differential-algebraic equations do not have exact solutions. Therefore, development of effective numerical techniques, which offer precise approximate solutions, has become an active research area. In this regard, Ibis and Bayram [12], Ibis et al. [13], Zurigat et al. [14] and Ding and Jiang [15] extended the application of Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), Fractional Differential Transform Method (FDTM), Homotopy Analysis Method (HAM) and waveform relaxation method to solve fractional order differential-algebraic equations.

The idea of Fourier to use arbitrary functions, even the one defined by different equations in its adjacent segments of its range, which a single analytical function cannot represent actually paved the way of using family of orthogonal functions. The processes having jump, delay and discontinuity may be better represented by orthogonal functions. The orthogonal triangular function sets developed by Deb et al. [16] are a complementary pair of piecewise linear polynomial function sets evolved from a simple dissection of block pulse function (BPF) set [17]. The authors have derived a complementary pair of operational matrices for first order integration in the TF domain and demonstrated that the TF domain technique for dynamical systems analysis is computationally more effective than the BPF domain technique. Besides system analysis, the orthogonal TFs also find applications in system identification, optimal controller design and numerical analysis of classical integral and differential equations [18, 19, 20, 21, 22]. Those successful applications have made us strongly believe that the TFs having enough potential to be applicable in fractional order
systems. To the best of our knowledge, there is no literature until date in fractional calculus that reported the use of orthogonal TFs for solving fractional order differential-algebraic equations. These facts motivated us to extend the application of orthogonal TFs to solve fractional order differential-algebraic equations shown in [3]. To accomplish our goal, we have proposed the generalized triangular function operational matrices for estimating the Riemann-Liouville fractional order integral in the TF domain. The rest of the paper is prepared as follows. Useful definitions of fractional calculus are provided in section 2. Generation of complementary pair of TF sets from BPF set is discussed in section 3. Section 4 presents the basic properties of orthogonal TF sets. The method of estimating classical and fractional integration in the TF domain is explained in section 5. An upper bound of absolute error between the exact Riemann-Liouville fractional order integral and its TF estimate is computed in section 6. Section 7 implements the proposed operational matrices on illustrative examples. Finally, the paper is concluded in section 8.

2. Basic definitions of fractional calculus

In this section, we provide widely used definitions of fractional calculus [23].

Definition 2.1 A real function \( f(t), \ t > 0 \), is said to be in the space \( C_\mu, \ \mu \in R \) if there exists a real number \( p (> \mu) \), such that \( f(t) = t^p f_1(t) \), where \( f_1(t) \in C[0, \infty) \) and it is said to be in the space \( C^{m}_\mu \) if and only if \( f^{(m)}(t) \in C_\mu, \ m \in N \).

Definition 2.2 The Riemann-Liouville fractional order integral of \( \alpha (> 0) \) of function \( f(t) \in C_\mu, \ \mu > -1 \) is defined as

\[
J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) \, d\tau
\]  

(4)

Definition 2.3 The Riemann-Liouville fractional order derivative (RL) of function \( f(t) \) is defined as

\[
D_t^\alpha f(t) = D^m J^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t - \tau)^{m-\alpha-1} f(\tau) \, d\tau, \ t > 0
\]  

(5)

where \( \alpha \) is a non-integer satisfying the relation \( m - 1 < \alpha \leq m, \ m \in N \).

Definition 2.4 The fractional order derivative of \( f(t) \) in Caputo sense is defined as

\[
D^\alpha_C f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t - \tau)^{m-\alpha-1} f^{(m)}(\tau) \, d\tau, \ t > 0
\]  

(6)

3. Triangular functions

In this section, firstly, we review block pulse functions in brief and then we introduce the method of dissecting the block pulse function set to formulate a complementary pair of orthogonal TF sets.

3.1 Review of block pulse functions

Let us consider a square integrable function \( f(t) \) of Lebesgue measure, which is continuous in the interval \([0, T]\). Divide the interval into \( m \) subintervals of constant width \( h = T/m \) as \([t_i, t_{i+1}) \), \( i = 0, 1, 2, \ldots, m - 1 \).
Let $\psi_m(t)$ be a set of block pulse functions containing $m$ component functions in the interval $[0, T)$.

$$\psi_m(t) = [\psi_0(t), \psi_1(t), \psi_2(t), \ldots, \psi_{m-1}(t)]^T$$

where $[\cdots]^T$ signifies transpose.

The $i^{th}$ component of the BPF vector $\psi_m(t)$ is defined as

$$\psi_i(t) = \begin{cases} 1 & ih \leq t < (i + 1)h \\ 0 & \text{otherwise} \end{cases} \quad i = 0, 1, 2, \ldots, m - 1$$

The square integrable function $f(t)$ can be approximated by BPFs as

$$f(t) \cong \sum_{i=0}^{m-1} f_i \psi_i(t) = [f_0, f_1, \ldots, f_{m-1}] \psi_m(t) = F^T \psi_m(t)$$
where the constant coefficients \( f_i \) are defined as \( f_i = \frac{1}{h} \int_{ih}^{(i+1)h} f(t) \, dt \).

The BPF estimate for first order integration of function \( f(t) \) can be derived as \[ 24 \]

\[
Jf(t) = \int_0^t f(s) \, ds \approx \int_0^t F^T \psi_m(s) \, ds = F^T \int_0^t \psi_m(s) \, ds = F^T P \psi_m(t) \quad (10)
\]

where \( P \) is the operational matrix for first order integration in the BPF domain:

\[
P = \frac{h}{2} \begin{bmatrix}
1 & 2 & 2 & \cdots & \cdots & 2 \\
0 & 1 & 2 & 2 & \cdots & 2 \\
0 & 0 & 1 & 2 & \cdots & 2 \\
\vdots & \vdots & \vdots & \ddots & \ddots & 2 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}_{m \times m}
\]

3.2 Complementary pair of TF sets

Let us divide the first component of BPF vector \( \psi_m(t) \) into a complementary pair of linear polynomial functions as shown in figure 1.

\[
\psi_0(t) = T_{10}(t) + T_{20}(t) \quad (11)
\]

where \( T_{10}(t) = (1 - \frac{t}{h}) \) and \( T_{20}(t) = (\frac{t}{h}) \)

and the second component \( \psi_1(t) \),

\[
\psi_1(t) = T_{11}(t) + T_{21}(t) \quad (12)
\]

where \( T_{11}(t) = 1 - (\frac{t-h}{h}) \) and \( T_{21}(t) = (\frac{t-h}{h}) \).

In the same fashion, we can divide the remaining components of \( \psi_m(t) \) into respective complementary pairs of linear polynomial functions. Thus, for the whole set of BPFs, we now have two sets of linear polynomial functions, namely, \( T_{1m}(t) \) and \( T_{2m}(t) \) each contains \( m \) component functions in the interval \([0, T] \).

\[
\psi_m(t) = T_{1m}(t) + T_{2m}(t) \quad (13)
\]

where \( T_{1m}(t) = [T_{10}(t), \ldots, T_{1m-1}(t)]^T \), \( T_{2m}(t) = [T_{20}(t), \ldots, T_{2m-1}(t)]^T \)

The triangular function vectors; \( T_{1m}(t) \) and \( T_{2m}(t) \) together form the entire set of BPFs, hence, \( T_{1m}(t) \) and \( T_{2m}(t) \) are complement to each other as far as BPF set is considered. We recognize from figure 1 that the shapes of \( T_{1i} \) and \( T_{2i} \) are left-handed and right-handed triangles, respectively. So, we name these two sets as left-handed triangular functions vector (LHTF) and right-handed triangular functions vector (RHTF), respectively.

Now we define the \( i^{th} \) component of the LHTF vector \( T_{1m}(t) \) as

\[
T_{1i}(t) = \begin{cases} 
1 - \left( \frac{t-ih}{h} \right) & \text{if } ih \leq t < (i+1)h, i = 0, 1, 2, \ldots, m - 1 \\
0 & \text{otherwise}
\end{cases}
\quad (14)
\]

and the \( i^{th} \) component of the RHTF vector \( T_{2m}(t) \) as

\[
T_{2i}(t) = \begin{cases} 
\frac{t-ih}{h} & \text{if } ih \leq t < (i+1)h, i = 0, 1, 2, \ldots, m - 1 \\
0 & \text{otherwise}
\end{cases}
\quad (15)
\]
Like BPFs, TFs can also be employed for the approximation of the square integrable function \( f(t) \) in the interval \([0, T]\).

\[
\begin{align*}
f(t) &\equiv [c_0, c_1, \ldots, c_{m-1}] T_{1m}(t) + [d_0, d_1, \ldots, d_{m-1}] T_{2m}(t) \\
&= C^T T_{1m}(t) + D^T T_{2m}(t)
\end{align*}
\]

where the constant coefficients \( c_i \) and \( d_i \) can be computed as \( c_i = f(ih), \ d_i = f((i+1)h) \).

The expressions for the coefficients \( c_i \) and \( d_i \) emphasize that the function evaluations at the equidistant nodes, \( t_i, i = 0, 1, 2, \ldots, m - 1 \) are enough to find their numerical values. Whereas the coefficients in BPF series representation in equation \[9\] demand the integration of \( f(t) \). Thus, the function approximation in TF domain is computationally more effective compared to that in BPF domain.

4. Basic properties of triangular functions

This section introduces orthogonal and a few operational properties of LHTF set \( T_{1m}(t) \) and RHTF set \( T_{2m}(t) \). The proofs of theorems \[4.1 \text{ to } 4.5\] are published elsewhere \[17\].

**Theorem 4.1.** If \( T_{1i}(t), T_{1j}(t) \in T_{1m}(t), \ i, j \leq m \), the condition of orthogonality of LHTF set \( T_{1m}(t) \) is \( \int_0^T T_{1i}(t) T_{1j}(t) \, dt = \begin{cases} \frac{h}{3} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \)

**Theorem 4.2.** If \( T_{2i}(t), T_{2j}(t) \in T_{2m}(t), \ i, j \leq m \), the condition of orthogonality of RHTF set \( T_{2m}(t) \) is \( \int_0^T T_{2i}(t) T_{2j}(t) \, dt = \begin{cases} \frac{h}{3} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \)

**Theorem 4.3.** If \( T_{1i}(t), T_{1j}(t) \in T_{1m}(t), \ i, j \leq m \), then
\[
T_{1i}(t) T_{1j}(t) = \begin{cases} 0 & \text{if } i \neq j \\ T_{1i}(t) & \text{if } i = j \end{cases}
\]

**Theorem 4.4.** If \( T_{2i}(t), T_{2j}(t) \in T_{2m}(t), \ i, j \leq m \), then
\[
T_{2i}(t) T_{2j}(t) = \begin{cases} 0 & \text{if } i \neq j \\ T_{2i}(t) & \text{if } i = j \end{cases}
\]

**Theorem 4.5.** If \( T_{1i}(t) \in T_{1m}(t) \) and \( T_{2j}(t) \in T_{2m}(t) \), then \( T_{1i}(t) T_{2j}(t) = 0, \ \forall i, j \leq m \)

**Theorem 4.6.** If \( g(t) \) is a square integrable function of Lebesgue measure and continuous in the interval \([0, T]\), then the TF estimate of \( n^{th} \) power of \( g(t) \) is \( (g(t))^n \equiv [c^n_0, c^n_1, \ldots, c^n_{m-1}] T_{1m}(t) + [d^n_0, d^n_1, \ldots, d^n_{m-1}] T_{2m}(t), n \in N. \)

**Proof.** Using equation \[16\] expanding the function \( g(t) \) in the orthogonal TF domain as
\[
g(t) \equiv C^T T_{1m}(t) + D^T T_{2m}(t)
\]

The product \( g(t) \times g(t) \) can be expressed in terms of complementary pair of TF sets as
\[
g(t) \times g(t) \equiv C^T \ast C^T (T_{1m}(t) T_{1m}(t)) + C^T \ast D^T (T_{1m}(t) T_{2m}(t)) + D^T \ast C^T (T_{2m}(t) T_{1m}(t)) + D^T \ast D^T (T_{2m}(t) T_{2m}(t))
\]

(18)
Here the operator \((\cdot, \cdot)\) denotes element-by-element product.
Employing theorems 4.3 to 4.5 the above equation can be simplified into the following form.

\[
g(t) \times g(t) \cong [c_0^2, c_1^2, \ldots, c_{m-1}^2] T1_m(t) + [d_0^2, d_1^2, \ldots, d_{m-1}^2] T2_m(t)
\]  

(19)

In the same manner, \(n^{th}\) power of \(g(t)\) can be estimated in the TF domain as shown in equation 20.

\[
(g(t))^n \cong [c_0^n, c_1^n, \ldots, c_{m-1}^n] T1_m(t) + [d_0^n, d_1^n, \ldots, d_{m-1}^n] T2_m(t)
\]  

(20)

**Theorem 4.7.** Suppose the square integrable functions; \(h_1(t), h_2(t), h_3(t), \ldots, h_n(t)\) are approximated via TFs. Then their product can be expressed as

\[
h_1(t) h_2(t) \cdots h_n(t) \cong \left[ c_0 h_1 c_2 \cdots c_n, \ldots, c_1 h_2 c_{m-1} \cdots c_n \right] T1_m(t) + \left[ d_0 h_1 d_2 \cdots d_n, \ldots, d_1 h_2 d_{m-1} \cdots d_n \right] T2_m(t)
\]

Proof. The piecewise linear approximation of \(h_1(t), h_2(t)\) and \(h_3(t)\) by TFs are

\[
\begin{align*}
h_1(t) & \cong C_1^T T1_m(t) + D_1^T T2_m(t) \\
h_2(t) & \cong C_2^T T1_m(t) + D_2^T T2_m(t) \\
h_3(t) & \cong C_3^T T1_m(t) + D_3^T T2_m(t)
\end{align*}
\]

(21)

Employing theorems 4.3 to 4.5 the product \(h_1(t) h_2(t)\) is estimated as

\[
h_1(t) h_2(t) \cong \left( C_1^T, C_2^T \right) T1_m(t) + \left( D_1^T, D_2^T \right) T2_m(t)
\]

(22)

\[
= \left[ c_0 h_1 c_2, c_1 h_2, \ldots, c_1 h_2 c_{m-1} \right] T1_m(t) + \left[ d_1 d_2, d_1, d_2, \ldots, d_1 d_{m-1} d_{m-1} \right] T2_m(t)
\]

The product \(h_1(t) h_2(t) h_3(t)\) in the TF domain is expressed as

\[
h_1(t) h_2(t) h_3(t) \cong \left[ c_0 h_1 c_2, c_3, c_1, c_2, \ldots, c_1 h_2 c_{m-1} c_{m-1} \right] T1_m(t) + \left[ d_1 d_2, d_3, d_1, d_2, \ldots, d_1 d_{m-1} \right] T2_m(t)
\]

(23)

Likewise, the product of \(n\) square integrable functions can be expressed as

\[
h_1(t) h_2(t) \cdots h_n(t) \cong \left[ c_1 h_2 c_3, \ldots, c_1 h_2 c_{m-1} c_{m-1} \right] T1_m(t) + \left[ d_1 d_2, \ldots, d_1 d_{m-1} d_{m-1} \right] T2_m(t)
\]

(24)

\[
\square
\]

5. Triangular Function Operational Matrices for Classical and Fractional Integration

In the following subsections, we establish the method of approximating classical as well as fractional integration in the TF domain.

5.1 The TF Estimate of First Order Integral of Function \(f(t)\)

One-fold integration of square integrable function \(f(t)\) is

\[
Jf(t) = \int_0^t f(s) \, ds
\]

(25)
Substituting the TF estimate of \( f(t) \) in equation [25] leads to

\[
\int_0^t f(s) \, ds \cong C^T \left[ \int_0^t T_{1m}(s) \, ds + D^T \int_0^t T_{2m}(s) \, ds \right] = C^T \left[ \int_0^t T_{10}(s) \, ds, \ldots, \int_0^t T_{1m-1}(s) \, ds \right]^T T_{1m}(t) + D^T \left[ \int_0^t T_{20}(s) \, ds, \ldots, \int_0^t T_{2m-1}(s) \, ds \right]^T T_{2m}(t)
\]

Therefore, the integration of function \( f(t) \) is now changed to the integration of LHTF set and RHTF set. Since the function \( f(t) \) is square integrable, its estimate is also square integrable.

The graph of \( T_{1i}(t) \) versus \( t \) and \( T_{2i}(t) \) versus \( t \) depicted in figure [1] can be expressed mathematically as

\[
T_{1i}(t) = u(t - ih) - \frac{t - ih}{h}u(t - ih) + \frac{t - (i + 1)h}{h}u(t - (i + 1)h)
\]

\[
T_{2i}(t) = \frac{t - ih}{h}u(t - ih) - \frac{t - (i + 1)h}{h}u(t - (i + 1)h) - u(t - (i + 1)h)
\]

We now integrate each component of LHTF set \( T_{1m}(t) \) using equation [27] and express the result in terms of LHTF set \( T_{1m}(t) \) and RHTF set \( T_{2m}(t) \).

\[
\int_0^t T_{10}(s) \, ds = \int_0^t \left( 1 - \frac{s}{h} \right) u(s) \, ds + \int_0^t \left( \frac{s - h}{h} \right) u(s - h) \, ds = \frac{1}{2} [0 \, 1 \, \ldots \, 1 \, 1] T_{1m}(t) + \frac{1}{2} [0 \, 1 \, \ldots \, 1 \, 1] T_{2m}(t)
\]

\[
\int_0^t T_{11}(s) \, ds = \frac{h}{2} [0 \, 0 \, 1 \, \ldots \, 1 \, 1] T_{1m}(t) + \frac{h}{2} [0 \, 1 \, \ldots \, 1 \, 1] T_{2m}(t)
\]

\[
\vdots
\]

\[
\int_0^t T_{1m-1}(s) \, ds = \frac{h}{2} [0 \, 0 \, 0 \, \ldots \, 0 \, 1] T_{1m}(t) + \frac{h}{2} [0 \, 0 \, \ldots \, 0 \, 1] T_{2m}(t)
\]

Therefore, the first order integration of LHTF set \( T_{1m}(t) \) is

\[
\int_0^t T_{1m}(s) \, ds = P_1 T_{1m}(t) + P_2 T_{2m}(t)
\]

where

\[
P_1 = \frac{h}{2} [0 \, 1 \, 1 \, \ldots \, 1], \quad P_2 = \frac{h}{2} [1 \, 1 \, 1 \, \ldots \, 1]
\]

Following the same procedure, the first order integration of RHTF set \( T_{2m}(t) \) using equation [28] is

\[
\int_0^t T_{2m}(s) \, ds = P_1 T_{1m}(t) + P_2 T_{2m}(t) = \int_0^t T_{1m}(s) \, ds
\]
Equation 25 becomes
\[ \int_0^t f(s) \, ds \approx (C^T P_1 + D^T P_1) T_{1_m}(t) + (C^T P_2 + D^T P_2) T_{2_m}(t) \]

(34)

Here \( P_1 \) and \( P_2 \) are complement to each other as far as \( P \) is considered. This complementary pair is acting as a first order integrator in the TF domain.

5.2 The TF Estimate of Riemann-Liouville Fractional Order Integral of Function \( f(t) \)

Replacing \( f(t) \) with its TF estimate in equation 4
\[
J^\alpha f(t) \approx C^T \begin{bmatrix} J^\alpha T_{1_0}(t) \\ J^\alpha T_{1_1}(t) \\ \vdots \\ J^\alpha T_{1_{m-1}}(t) \end{bmatrix} + D^T \begin{bmatrix} J^\alpha T_{2_0}(t) \\ J^\alpha T_{2_1}(t) \\ \vdots \\ J^\alpha T_{2_{m-1}}(t) \end{bmatrix}
\]

(35)

Similar to equation 29, we compute the \( \alpha \) - order Riemann-Liouville fractional integral of \( T_{1_0}(t) \) and express the result by means of complementary pair of TF sets.

\[
J^\alpha T_{1_0}(t) = \frac{h^\alpha}{\Gamma(\alpha+2)} \begin{bmatrix} 0 & \varsigma_1 & \varsigma_2 & \cdots & \varsigma_{m-1} \\ \varsigma_1 & \varsigma_2 & \varsigma_3 & \cdots & \varsigma_m \end{bmatrix} T_{1_m}(t) + \begin{bmatrix} \varsigma_1 \\ \varsigma_2 \\ \vdots \\ \varsigma_{m-1} \end{bmatrix} T_{2_m}(t)
\]

(36)

where \( \varsigma_j = (j^\alpha (1 + \alpha - j) + (j-1)^{\alpha+1}) \), \( j = 1, 2, \ldots, m \).

\[
J^\alpha T_{1_1}(t) = \frac{h^\alpha}{\Gamma(\alpha+2)} \begin{bmatrix} 0 & 0 & \varsigma_1 & \cdots & \varsigma_{m-2} \\ 0 & \varsigma_1 & \varsigma_2 & \cdots & \varsigma_{m-1} \end{bmatrix} T_{1_m}(t) + \begin{bmatrix} \varsigma_1 \\ \varsigma_2 \\ \vdots \\ \varsigma_{m-1} \end{bmatrix} T_{2_m}(t)
\]

(37)

\[
\vdots
\]

\[
J^\alpha T_{1_{m-1}}(t) = \frac{h^\alpha}{\Gamma(\alpha+2)} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \varsigma_1 & \cdots & 0 \end{bmatrix} T_{1_m}(t) + \begin{bmatrix} \varsigma_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} T_{2_m}(t)
\]

(38)

Therefore, the Riemann-Liouville fractional integral of order \( \alpha \) of LHTF set \( T_{1_m}(t) \) is
\[
\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} T_{1_m}(\tau) \, d\tau = P_1^\alpha T_{1_m}(t) + P_2^\alpha T_{2_m}(t)
\]

(39)

where
\[
P_1^\alpha = \frac{h^\alpha}{\Gamma(\alpha+2)} \begin{bmatrix} 0 & \varsigma_1 & \varsigma_2 & \cdots & \varsigma_{m-1} \\ 0 & 0 & \varsigma_1 & \cdots & \varsigma_{m-2} \\ 0 & 0 & 0 & \varsigma_1 & \cdots \end{bmatrix}, \quad P_2^\alpha = \frac{h^\alpha}{\Gamma(\alpha+2)} \begin{bmatrix} \varsigma_1 \\ 0 & \varsigma_1 & \varsigma_2 & \cdots & \varsigma_{m-1} \\ 0 & 0 & \varsigma_1 & \cdots & \varsigma_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \varsigma_1 & \cdots \\ 0 & 0 & 0 & \cdots & \varsigma_1 \end{bmatrix}
\]
Let us denote the TF estimate of operational matrices is a particular case of generalized operational matrices.

The TF estimate of fractional order integral is reduced to the TF estimate of first order integral when \( \alpha = 1 \). Therefore, the classical complementary pair of operational matrices is a particular case of generalized operational matrices.

### 6. Error analysis

Let us denote the TF estimate of function \( f(t) \) as

\[
f_{TF}(t) = C^T \Phi(t) + D^T \Psi(t)
\]

where

\[
P^\alpha = \frac{h^\alpha}{\Gamma(\alpha+2)}
\]

From equations 35, 39 and 40

\[
J_{TF}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau
\]

\[
\cong \left( C^T P^\alpha_1 + D^T P^\alpha_3 \right) \Phi(t) + \left( C^T P^\alpha_2 + D^T P^\alpha_4 \right) \Psi(t)
\]

For the special case of \( \alpha = 1 \),

\[
P^1 = P^\alpha_3 = P_1, P^\alpha_2 = P^\alpha_4 = P_2
\]

The TF estimate of fractional order integral is reduced to the TF estimate of first order integral when \( \alpha = 1 \). Therefore, the classical complementary pair of operational matrices is a particular case of generalized operational matrices.

Expanding the exact function \( f(t) \) by Taylor series with the center \( ih \) as

\[
f(t) = f(ih) + (t-ih) f'(ih) + \frac{(t-ih)^2}{2} f''(ih) + \sum_{k=3}^{\infty} \frac{(t-ih)^k}{k!} f^{(k)}(ih)
\]

From equations 44 and 45 the absolute error between the function and its TF estimate can be determined as

\[
|f(t) - f_{TF}(t)| = \frac{(t-ih)^2}{2} |f''(ih)| + O(t-ih)^3
\]

Because \( (t-ih) < h \) and \( mh = T \), the above equation becomes

\[
|f(t) - f_{TF}(t)| \leq \frac{T^2}{2m^2} |f''(ih)| + O\left( \frac{1}{m^3} \right)
\]
We replace $f(t)$ with $f_{TF}(t)$ in equation (41) and we call the resulting integral the $m^{th}$ approximate of the $\alpha$-order Riemann-Liouville fractional integral of $f(t)$.

$$J^\alpha_{TF}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_{TF}(\tau) \, d\tau$$

The absolute error between the exact fractional integral $J^\alpha f(t)$ and the $m^{th}$ approximate $J^\alpha_{TF}f(t)$ is

$$\epsilon_m = |J^\alpha f(t) - J^\alpha_{TF}f(t)|$$

$$= \frac{1}{\Gamma(\alpha)} \left[ \sum_{r=0}^{\lfloor T \rfloor} (t-\tau)^{\alpha-1} |f(\tau) - f_{TF}(\tau)| \, d\tau \right] + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |f(\tau) - f_{TF}(\tau)| \, d\tau$$

$$\leq \frac{1}{\Gamma(\alpha)} \left[ \sum_{r=0}^{\lfloor T \rfloor} \left( T^2 \frac{1}{2m^2} |f''(ih)| + O\left( \frac{1}{m^3} \right) \right) \right] + \frac{1}{\Gamma(\alpha)} \int_0^T (t-\tau)^{\alpha-1} \left( T^2 \frac{1}{2m^2} |f''(ih)| + O\left( \frac{1}{m^3} \right) \right) d\tau$$

$$\leq \frac{1}{\Gamma(\alpha)} \left( T^2 \frac{1}{2m^2} |f''(ih)| + O\left( \frac{1}{m^3} \right) \right) T^\alpha + O\left( \frac{1}{m^3} \right)$$

(49)

We now consider the following assumption.

$$Max |f''(ih)| \leq M, \quad i = 0, 1, 2, \ldots, m - 1$$

(50)

where $M$ is finite positive value.

From equations (49) and (50) the upper bound of absolute error between $J^\alpha f(t)$ and $J^\alpha_{TF}f(t)$ can be estimated as

$$\epsilon_m \leq \frac{MT^{2+\alpha}}{2m^2\Gamma(\alpha + 1)} + O\left( \frac{1}{m^3} \right)$$

(51)

To confirm whether the maximal absolute error caused by TFs is smaller than or equal to the theoretical upper bound derived in equation (51) we consider the function $f(t) = t$ in the interval $[0, 1]$, which is divided into five equal subintervals $m = 5$. We select the value of fractional order as 0.5.

The exact fractional integral of function $f(t)$ is

$$J^{0.5}t = \frac{\Gamma(2)}{\Gamma(2.5)} t^{1.5}$$

(52)

Using equation (51) the TF estimate is obtained as

$$J^{0.5}_{TF} \approx \begin{bmatrix} 0 \\ 0.0672835339205376 \\ 0.190306572389629 \\ 0.349615497789465 \\ 0.538268271364301 \end{bmatrix}^T T^{1.5} + \begin{bmatrix} 0.0672835339205376 \\ 0.190306572389629 \\ 0.349615497789465 \\ 0.538268271364301 \\ 0.752252778063675 \end{bmatrix}^T T^{2m(t)}$$

(53)
In the BPF domain \cite{25},

\[
\begin{pmatrix}
0.03364176669602688 \\
0.128790563155083 \\
0.269961035089547 \\
0.443941884576883 \\
0.645260524713988 \\
0.870557367621520
\end{pmatrix}
\]

\[
J_{BPF}^{0.5} \cong \begin{bmatrix}
\alpha_1 & 0 & 0 \\
0 & \alpha_2 & 0 \\
0 & 0 & \alpha_3 \\
\end{bmatrix} T
\]

\[
\psi_m(t)
\]

Table 1 presents the absolute error given by TF domain analysis and BPF domain analysis. It can be noticed that the TF solution and the samples of exact fractional integral are precisely equal that demonstrates the accuracy and efficiency of the proposed TF approximation of Riemann-Liouville fractional order integral. For the selected step size, BPFs could not offer approximate solution with reasonable accuracy due to its piecewise constant nature.

7. Applications and results

In this section, we solve linear and nonlinear fractional order differential-algebraic equations by employing the basic properties of orthogonal TFs and the derived generalized triangular function operational matrices. In the following examples, the Matlab built-in function ’fsolve’ is used for solving system of algebraic equations.

Example 1

Let us consider the following linear fractional order differential-algebraic equation \cite{15}

\[
\begin{align*}
D^{1/2}x_1(t) + 2x_1(t) - \frac{\Gamma(7/2)}{\Gamma(3)}x_2(t) + x_3(t) &= f_1(t), \\
D^{1/2}x_2(t) + x_2(t) + x_3(t) &= f_2(t), \\
2x_1(t) + x_2(t) - x_3(t) &= f_3(t),
\end{align*}
\]

with initial conditions \(x_1(0) = x_2(0) = x_3(0) = 0\).

where \(f_1(t) = 2t^{5/2} + \sin t\), \(f_2(t) = \frac{\Gamma(7/2)}{\Gamma(5/2)}t^{3/2} + t^2 + \sin t\) and \(f_3(t) = 2t^{5/2} + t^2 - \sin t\).

The exact solution is \(x_1(t) = t^{5/2}, x_2(t) = t^2, x_3(t) = \sin t\).

Solution:

Performing fractional integration on both sides of equation \cite{55} results in the following form.

\[
\begin{align*}
x_1(t) &= J^{1/2}(f_1(t)) - J^{1/2}(2x_1(t)) + \frac{\Gamma(7/2)}{\Gamma(3)}J^{1/2}(x_2(t)) - J^{1/2}(x_3(t)), \\
x_2(t) &= J^{1/2}(f_2(t)) - J^{1/2}(x_2(t)) - J^{1/2}(x_3(t)), \\
x_3(t) &= -f_3(t) + 2x_1(t) + x_2(t)
\end{align*}
\]

Expanding the known functions; \(f_1(t), f_2(t)\) and \(f_3(t)\) and the unknown functions; \(x_1(t), x_2(t)\) and \(x_3(t)\) by means of TFs,

\[
\begin{align*}
f_1(t) &\cong C_{01}^T T_{1m}(t) + D_{01}^T T_{2m}(t), \quad f_2(t) \cong C_{02}^T T_{1m}(t) + D_{02}^T T_{2m}(t)
\end{align*}
\]

\[
\begin{align*}
f_3(t) &\cong C_{03}^T T_{1m}(t) + D_{03}^T T_{2m}(t), \quad x_1(t) \cong C_{1}^T T_{1m}(t) + D_{1}^T T_{2m}(t)
\end{align*}
\]

\[
\begin{align*}
x_2(t) &\cong C_{2}^T T_{1m}(t) + D_{2}^T T_{2m}(t), \quad x_3(t) \cong C_{3}^T T_{1m}(t) + D_{3}^T T_{2m}(t)
\end{align*}
\]
By substituting equations 41 and 57 in 56 and comparing the coefficients of LHTF and RHTF set of resulting equation, we have the following expressions.

\[
\begin{align*}
C_1^T &= \left( C_{01}^T P_1^{1/2} + D_{01}^T P_3^{1/2} \right) - 2 \left( C_1^T P_1^{1/2} + D_1^T P_3^{1/2} \right) + \frac{\Gamma(7/2)}{\Gamma(3)} C_1^T P_1^{1/2} + \frac{\Gamma(7/2)}{\Gamma(3)} D_1^T P_3^{1/2} \\
D_1^T &= \left( C_{01}^T P_2^{1/2} + D_{01}^T P_4^{1/2} \right) - 2 \left( C_1^T P_2^{1/2} + D_1^T P_4^{1/2} \right) + \frac{\Gamma(7/2)}{\Gamma(3)} C_1^T P_2^{1/2} + \frac{\Gamma(7/2)}{\Gamma(3)} D_1^T P_4^{1/2} \\
C_2^T &= \left( C_{02}^T P_1^{1/2} + D_{02}^T P_3^{1/2} \right) - \left( C_2^T P_1^{1/2} + D_2^T P_3^{1/2} \right) - \left( C_2^T P_1^{1/2} + D_2^T P_3^{1/2} \right) \\
D_2^T &= \left( C_{02}^T P_2^{1/2} + D_{02}^T P_4^{1/2} \right) - \left( C_2^T P_2^{1/2} + D_2^T P_4^{1/2} \right) - \left( C_2^T P_2^{1/2} + D_2^T P_4^{1/2} \right) \\
C_3^T &= -C_{03}^T + 2C_1^T + C_2^T \\
D_3^T &= -D_{03}^T + 2D_1^T + D_2^T \\
\end{align*}
\]

(58)

Upon solving the above system of linear algebraic equations, the approximate solution of \( x_1(t), x_2(t) \) and \( x_3(t) \) are obtained. The maximal absolute errors produced by TFs for various values of \( h \) are calculated and tabulated in Table 2. The time taken by TFs for computing approximate solutions in each run is recorded and shown in Table 2. As the number of subintervals is increasing, accuracy is increasing, hence, the approximate solutions are converging to the exact solutions.

**Table 1. Absolute error using TFs and BPFs**

| \( t \) | \( |J^\alpha f(t) - J_{TF}^\alpha f(t)| \) | \( |J^\alpha f(t) - J_{BPF}^\alpha f(t)| \) |
|---|---|---|
| 0 | 0 | 0.0336417669602 |
| 0.2 | 0 | 0.0615115192345 |
| 0.4 | 0 | 0.0796544626999 |
| 0.6 | 0 | 0.0943263867874 |
| 0.8 | 0 | 0.106992253349 |
| 1 | 0 | 0.118304589557 |

**Table 2. Absolute error using TFs for example 1**

<table>
<thead>
<tr>
<th>( h )</th>
<th>Maximal absolute error</th>
<th>CPU time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1(t) )</td>
<td>( x_2(t) )</td>
<td>( x_3(t) )</td>
</tr>
<tr>
<td>0.1</td>
<td>5.7540940e-04</td>
<td>7.42963900e-04</td>
</tr>
<tr>
<td>0.02</td>
<td>0.04528065e-05</td>
<td>4.13615672e-05</td>
</tr>
<tr>
<td>0.01</td>
<td>0.06590342e-06</td>
<td>1.11465507e-05</td>
</tr>
<tr>
<td>0.002</td>
<td>6.66389858e-07</td>
<td>7.44211153e-07</td>
</tr>
</tbody>
</table>

**Example 2**

Consider the nonlinear fractional order differential-algebraic equation

\[
\begin{align*}
D^\alpha x(t) - x(t) + z(t)x(t) &= 1, \\
D^\alpha z(t) - y(t) + x^3(t) + z(t) &= 0, \\
y(t) - x^2(t) &= 0, \quad t \in [0, 1], \quad 0 < \alpha \leq 1
\end{align*}
\]

(59)
subject to initial conditions $x(0) = y(0) = z(0) = 1$.

For the special case of $\alpha = 1$, the given problem has analytical solution; $x(t) = e^t, y(t) = e^{2t}, z(t) = e^{-t}$.

**Solution:**

The given problem can be simplified as

\[
\begin{align*}
D_t^\alpha x(t) &= 1 + x(t) - z(t) x(t) \\
D_t^\alpha z(t) &= -z(t)
\end{align*}
\]  

Multiplying with fractional integrator $J^\alpha$ on both sides of above equation gives

\[
\begin{align*}
x(t) &= x(0) + J^\alpha (1) + J^\alpha (x(t)) - J^\alpha (z(t) x(t)) \\
z(t) &= z(0) - J^\alpha (z(t))
\end{align*}
\]  

In the TF domain, the approximations for $x(0), x(t)$ and $z(t)$ are attained as follows.

\[
\begin{align*}
1 &\approx C_1^T T_1^m(t) + D_0^T T^2_2(t) \\
x(t) &\approx C_1^T T_1^m(t) + D_1^T T^2_2(t) \\
z(t) &\approx C_2^T T_1^m(t) + D_2^T T^2_2(t)
\end{align*}
\]  

From theorem 4.7 and equations 41, 61 and 62 we get the following system of nonlinear algebraic equations.

\[
\begin{align*}
C_1^T &= C_0^T + (C_0^T P_1^a + D_0^T P_3^a) + (C_1^T P_1^a + D_1^T P_3^a) - (C_1^T \cdot C_2^T) P_1^a - \\
&\quad (D_1^T \cdot D_2^T) P_3^a \\
D_1^T &= D_0^T + (C_0^T P_2^a + D_0^T P_4^a) + (C_1^T P_2^a + D_1^T P_4^a) - (C_1^T \cdot C_2^T) P_2^a - \\
&\quad (D_1^T \cdot D_2^T) P_4^a \\
C_2^T &= C_0^T - (C_2^T P_1^a + D_2^T P_3^a) \\
D_2^T &= D_0^T - (C_2^T P_2^a + D_2^T P_4^a)
\end{align*}
\]  

The above system of nonlinear algebraic equations are solved using step size of 0.002 and the approximate solutions of $x(t)$ and $z(t)$ for different values of $\alpha$ are given in tables 3 to 8. The TF solutions are in good agreement with the solutions obtained by FDTM [13], ADM [12], VIM [12] and HAM [13].
Example 3
Consider the following nonlinear fractional order differential-algebraic equation
\[
\begin{align*}
D_{\alpha}^2 y(t) &= y(t) - z(t) w(t) + \sin t + t \cos t, \\
D_{\alpha}^2 z(t) &= t w(t) + y^2(t) + \sec^2 t - t^2 (\cos t + \sin^2 t), \\
0 &= y(t) - w(t) + t (\cos t - \sin t), t \in [0,1], 0 < \alpha \leq 1
\end{align*}
\]  
(64)

with initial conditions \( y(0) = z(0) = w(0) = 0 \).
In case of \( \alpha = 1 \), we have analytical solution: \( y(t) = t \sin t, z(t) = \tan t, w(t) = t \cos t \).

Solution:
Simplifying equation 64
\[
\begin{align*}
D_{\alpha}^2 y(t) &= y(t) - z(t) y(t) - z(t) f_1(t) + f_2(t) \\
D_{\alpha}^2 z(t) &= ty(t) + y^2(t) + f_3(t)
\end{align*}
\]  
(65)

where \( f_1(t) = t \cos t - t \sin t, f_2(t) = \sin t + t \cos t \) and \( f_3(t) = \sec^2 t - t^2 \sin t - t^2 \sin^2 t \).
Rewriting equation 65
\[
\begin{align*}
y(t) &= J^\alpha (y(t)) - J^\alpha (z(t) y(t)) - J^\alpha (z(t) f_1(t)) + J^\alpha (f_2(t)) \\
z(t) &= J^\alpha (ty(t)) + J^\alpha (y^2(t)) + J^\alpha (f_3(t))
\end{align*}
\]  
(66)

Table 4. Approximate solution of \( x(t) \) for \( \alpha = 0.75 \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>FDTM</th>
<th>ADM</th>
<th>VIM</th>
<th>HAM</th>
<th>TF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>1.2187069</td>
<td>1.2187068</td>
<td>1.2187068</td>
<td>1.2187068</td>
<td>1.2187043</td>
</tr>
<tr>
<td>0.2</td>
<td>1.4000280</td>
<td>1.4000278</td>
<td>1.4000279</td>
<td>1.4000279</td>
<td>1.4000262</td>
</tr>
<tr>
<td>0.3</td>
<td>1.5841270</td>
<td>1.5841268</td>
<td>1.5841268</td>
<td>1.5841270</td>
<td>1.5841259</td>
</tr>
<tr>
<td>0.4</td>
<td>1.7769089</td>
<td>1.7769086</td>
<td>1.7769086</td>
<td>1.7769089</td>
<td>1.7769081</td>
</tr>
<tr>
<td>0.5</td>
<td>1.9813870</td>
<td>1.9813865</td>
<td>1.9813865</td>
<td>1.9813869</td>
<td>1.9813866</td>
</tr>
<tr>
<td>0.6</td>
<td>2.1997453</td>
<td>2.1997446</td>
<td>2.1997447</td>
<td>2.1997452</td>
<td>2.1997453</td>
</tr>
<tr>
<td>0.7</td>
<td>2.4338838</td>
<td>2.4338829</td>
<td>2.4338829</td>
<td>2.4338838</td>
<td>2.4338842</td>
</tr>
<tr>
<td>0.8</td>
<td>2.6856249</td>
<td>2.6856238</td>
<td>2.6856234</td>
<td>2.6856248</td>
<td>2.6856256</td>
</tr>
<tr>
<td>0.9</td>
<td>2.9568125</td>
<td>2.9568110</td>
<td>2.9568095</td>
<td>2.9568130</td>
<td>2.9568136</td>
</tr>
<tr>
<td>1</td>
<td>3.2493684</td>
<td>3.2493666</td>
<td>3.2493623</td>
<td>3.2493749</td>
<td>3.2493700</td>
</tr>
</tbody>
</table>

The following equation gives the TF estimates of \( f_1(t), f_2(t), f_3(t), t, y(t) \) and \( z(t) \).
\[
\begin{align*}
f_1(t) &\approx C_{10}^T T_{11}(t) + D_{11}^T T_{21}(t), f_2(t) \approx C_{20}^T T_{11}(t) + D_{21}^T T_{21}(t) \\
f_3(t) &\approx C_{30}^T T_{11}(t) + D_{30}^T T_{21}(t), t \approx C_{40}^T T_{11}(t) + D_{40}^T T_{21}(t) \\
y(t) &\approx C_{11}^T T_{11}(t) + D_{11}^T T_{22}(t), z(t) \approx C_{21}^T T_{11}(t) + D_{21}^T T_{22}(t)
\end{align*}
\]  
(67)
Figure 2. Comparison between TF solution and exact solution for $\alpha = 1$

Figure 3. TF solutions for example 3
Table 5. Approximate solution of $x(t)$ for $\alpha = 0.5$

<table>
<thead>
<tr>
<th>$t$</th>
<th>FDTM</th>
<th>ADM</th>
<th>VIM</th>
<th>HAM</th>
<th>TF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.4678849</td>
<td>1.4678849</td>
<td>1.4678849</td>
<td>1.4678849</td>
<td>1.4678635</td>
</tr>
<tr>
<td>0.1</td>
<td>1.7411322</td>
<td>1.7411321</td>
<td>1.7411320</td>
<td>1.7411321</td>
<td>1.7411215</td>
</tr>
<tr>
<td>0.2</td>
<td>1.9927891</td>
<td>1.9927891</td>
<td>1.9927879</td>
<td>1.9927891</td>
<td>1.9927834</td>
</tr>
<tr>
<td>0.3</td>
<td>2.2392557</td>
<td>2.2392557</td>
<td>2.2392487</td>
<td>2.2392557</td>
<td>2.2392532</td>
</tr>
<tr>
<td>0.4</td>
<td>2.4871415</td>
<td>2.4871415</td>
<td>2.4871134</td>
<td>2.4871415</td>
<td>2.4871415</td>
</tr>
<tr>
<td>0.5</td>
<td>2.7401183</td>
<td>2.7401183</td>
<td>2.7400292</td>
<td>2.7401183</td>
<td>2.7401203</td>
</tr>
<tr>
<td>0.6</td>
<td>3.0006469</td>
<td>3.0006469</td>
<td>3.0004093</td>
<td>3.0006469</td>
<td>3.0006508</td>
</tr>
<tr>
<td>0.7</td>
<td>3.2706054</td>
<td>3.2706053</td>
<td>3.2700447</td>
<td>3.2706053</td>
<td>3.2706109</td>
</tr>
<tr>
<td>0.8</td>
<td>3.5515666</td>
<td>3.5515664</td>
<td>3.5503632</td>
<td>3.5515665</td>
<td>3.5515737</td>
</tr>
<tr>
<td>0.9</td>
<td>3.8450346</td>
<td>3.8449407</td>
<td>3.8455419</td>
<td>3.8450350</td>
<td>3.8449494</td>
</tr>
<tr>
<td>1</td>
<td>3.8450346</td>
<td>3.8449407</td>
<td>3.8455419</td>
<td>3.8450350</td>
<td>3.8449494</td>
</tr>
</tbody>
</table>

Table 6. Approximate solution of $z(t)$ for $\alpha = 1$

<table>
<thead>
<tr>
<th>$t$</th>
<th>FDTM</th>
<th>ADM</th>
<th>VIM</th>
<th>HAM</th>
<th>TF</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9048374</td>
<td>0.9048374</td>
<td>0.9048374</td>
<td>0.9048373</td>
<td>0.9048374</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.8187308</td>
<td>0.8187307</td>
<td>0.8187307</td>
<td>0.8187306</td>
<td>0.8187307</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.7408182</td>
<td>0.7408182</td>
<td>0.7408182</td>
<td>0.7408182</td>
<td>0.7408182</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.6703201</td>
<td>0.6703200</td>
<td>0.6703200</td>
<td>0.6703199</td>
<td>0.6703200</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.6065307</td>
<td>0.6065306</td>
<td>0.6065306</td>
<td>0.6065305</td>
<td>0.6065306</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.5488116</td>
<td>0.5488116</td>
<td>0.5488116</td>
<td>0.5488115</td>
<td>0.5488116</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>0.4965853</td>
<td>0.4965853</td>
<td>0.4965853</td>
<td>0.4965853</td>
<td>0.4965853</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.4493290</td>
<td>0.4493289</td>
<td>0.4493289</td>
<td>0.4493288</td>
<td>0.4493289</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>0.4065697</td>
<td>0.4065696</td>
<td>0.4065696</td>
<td>0.4065696</td>
<td>0.4065696</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.3678794</td>
<td>0.3678794</td>
<td>0.3678794</td>
<td>0.3678794</td>
<td>0.3678794</td>
<td></td>
</tr>
</tbody>
</table>

Table 7. Approximate solution of $z(t)$ for $\alpha = 0.75$

<table>
<thead>
<tr>
<th>$t$</th>
<th>FDTM</th>
<th>ADM</th>
<th>VIM</th>
<th>HAM</th>
<th>TF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.8282505</td>
<td>0.8282506</td>
<td>0.8282505</td>
<td>0.8282505</td>
<td>0.8282476</td>
</tr>
<tr>
<td>0.2</td>
<td>0.7325847</td>
<td>0.7325847</td>
<td>0.7325847</td>
<td>0.7325847</td>
<td>0.7325825</td>
</tr>
<tr>
<td>0.3</td>
<td>0.6603375</td>
<td>0.6603375</td>
<td>0.6603375</td>
<td>0.6603374</td>
<td>0.6603357</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6021211</td>
<td>0.6021211</td>
<td>0.6021211</td>
<td>0.6021210</td>
<td>0.6021196</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5360262</td>
<td>0.5360262</td>
<td>0.5360262</td>
<td>0.5360255</td>
<td>0.5360126</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5128511</td>
<td>0.5128511</td>
<td>0.5128511</td>
<td>0.5128505</td>
<td>0.5128393</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4765549</td>
<td>0.4765549</td>
<td>0.4765549</td>
<td>0.4765549</td>
<td>0.4765539</td>
</tr>
<tr>
<td>0.8</td>
<td>0.4452924</td>
<td>0.4452926</td>
<td>0.4452945</td>
<td>0.4452924</td>
<td>0.4452915</td>
</tr>
<tr>
<td>0.9</td>
<td>0.4176821</td>
<td>0.4176823</td>
<td>0.4452945</td>
<td>0.4176820</td>
<td>0.4176812</td>
</tr>
<tr>
<td>1</td>
<td>0.3931083</td>
<td>0.3931086</td>
<td>0.3931086</td>
<td>0.3931083</td>
<td>0.3931076</td>
</tr>
</tbody>
</table>

We now substitute equations [41] and [67] in [66] and equate the coefficients of LHTF and RHTF sets on both sides of resulting equation.

\[
\begin{align*}
C_1^T &= (C_{12}^T P_{12}^2 + D_{12}^T P_{12}^3) + (C_{13}^T P_{13}^2 + D_{13}^T P_{13}^3) - (C_{11}^T P_{11}^2 - (D_{11}^T P_{11}^3) - (C_{10}^T P_{10}^2 - (D_{10}^T P_{10}^3) - (C_{19}^T P_{19}^2 - (D_{19}^T P_{19}^3))

D_1^T &= (C_{22}^T P_{22}^2 + D_{22}^T P_{22}^3) + (C_{23}^T P_{23}^2 + D_{23}^T P_{23}^3) - (C_{21}^T P_{21}^2 - (D_{21}^T P_{21}^3) - (C_{20}^T P_{20}^2 - (D_{20}^T P_{20}^3) - (C_{19}^T P_{19}^2 - (D_{19}^T P_{19}^3))

C_2^T &= (C_{32}^T P_{32}^2 + D_{32}^T P_{32}^3) + (C_{42}^T P_{42}^2 + D_{42}^T P_{42}^3) + (C_{52}^T P_{52}^2 + D_{52}^T P_{52}^3)

D_2^T &= (C_{62}^T P_{62}^2 + D_{62}^T P_{62}^3) + (C_{72}^T P_{72}^2 + D_{72}^T P_{72}^3) + (C_{82}^T P_{82}^2 + D_{82}^T P_{82}^3) + (C_{92}^T P_{92}^2 + D_{92}^T P_{92}^3) + (C_{10}^T P_{10}^2 + D_{10}^T P_{10}^3)

\end{align*}
\]

(68)
Table 8. Approximate solution of $z(t)$ for $\alpha = 0.5$

<table>
<thead>
<tr>
<th>$t$</th>
<th>FDTM</th>
<th>ADM</th>
<th>VIM</th>
<th>HAM</th>
<th>TF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.723578</td>
<td>0.723578</td>
<td>0.723578</td>
<td>0.723578</td>
<td>0.7235556</td>
</tr>
<tr>
<td>0.1</td>
<td>0.6437883</td>
<td>0.6437882</td>
<td>0.6437886</td>
<td>0.6437882</td>
<td>0.6437750</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5920184</td>
<td>0.5920184</td>
<td>0.5920222</td>
<td>0.5920184</td>
<td>0.59202009</td>
</tr>
<tr>
<td>0.3</td>
<td>0.5536063</td>
<td>0.5536062</td>
<td>0.5536241</td>
<td>0.5536062</td>
<td>0.5535990</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5231566</td>
<td>0.5231565</td>
<td>0.5232163</td>
<td>0.5231565</td>
<td>0.5231507</td>
</tr>
<tr>
<td>0.5</td>
<td>0.4980246</td>
<td>0.4980245</td>
<td>0.4981839</td>
<td>0.4980245</td>
<td>0.4980196</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4767027</td>
<td>0.4767027</td>
<td>0.4770678</td>
<td>0.4767027</td>
<td>0.4766985</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4582460</td>
<td>0.4582460</td>
<td>0.4589938</td>
<td>0.4582460</td>
<td>0.4582423</td>
</tr>
<tr>
<td>0.8</td>
<td>0.4420214</td>
<td>0.4420214</td>
<td>0.4424279</td>
<td>0.4420214</td>
<td>0.4420181</td>
</tr>
<tr>
<td>0.9</td>
<td>0.4275836</td>
<td>0.4275835</td>
<td>0.4270570</td>
<td>0.4275835</td>
<td>0.4275806</td>
</tr>
<tr>
<td>1</td>
<td>0.4148207</td>
<td>1.0148207</td>
<td>0.9148207</td>
<td>0.9148207</td>
<td>0.9148208</td>
</tr>
</tbody>
</table>

Table 9. Approximate solution of $x(t)$ for $\alpha = 1$

<table>
<thead>
<tr>
<th>$t$</th>
<th>FDTM</th>
<th>ADM</th>
<th>VIM</th>
<th>HAM</th>
<th>TF</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.8492996</td>
<td>0.8492996</td>
<td>0.8492996</td>
<td>0.8492996</td>
<td>0.8492995</td>
<td>0.8492966</td>
</tr>
<tr>
<td>0.1</td>
<td>0.8016697</td>
<td>0.8016697</td>
<td>0.8016697</td>
<td>0.8016697</td>
<td>0.8016696</td>
<td>0.8016674</td>
</tr>
<tr>
<td>0.2</td>
<td>0.7978999</td>
<td>0.7978998</td>
<td>0.7978998</td>
<td>0.7978998</td>
<td>0.7979000</td>
<td>0.7978979</td>
</tr>
<tr>
<td>0.3</td>
<td>0.8250871</td>
<td>0.8250871</td>
<td>0.8250871</td>
<td>0.8250871</td>
<td>0.8250873</td>
<td>0.8250855</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8760144</td>
<td>0.8760143</td>
<td>0.8760143</td>
<td>0.8760143</td>
<td>0.8760146</td>
<td>0.8760129</td>
</tr>
<tr>
<td>0.5</td>
<td>0.9454582</td>
<td>0.9454581</td>
<td>0.9454581</td>
<td>0.9454581</td>
<td>0.9454582</td>
<td>0.9454568</td>
</tr>
<tr>
<td>0.6</td>
<td>1.0290757</td>
<td>1.0290756</td>
<td>1.0290756</td>
<td>1.0290756</td>
<td>1.0290757</td>
<td>1.0290744</td>
</tr>
<tr>
<td>0.7</td>
<td>1.1229595</td>
<td>1.1229594</td>
<td>1.1229594</td>
<td>1.1229594</td>
<td>1.1229592</td>
<td>1.1229582</td>
</tr>
<tr>
<td>0.8</td>
<td>1.2234368</td>
<td>1.2234365</td>
<td>1.2234365</td>
<td>1.2234365</td>
<td>1.2234363</td>
<td>1.2234354</td>
</tr>
<tr>
<td>0.9</td>
<td>1.3269767</td>
<td>1.3269759</td>
<td>1.3269759</td>
<td>1.3269759</td>
<td>1.3269757</td>
<td>1.3269747</td>
</tr>
<tr>
<td>1</td>
<td>1.2093504</td>
<td>1.2093504</td>
<td>1.2093504</td>
<td>1.2093504</td>
<td>1.2093498</td>
<td>1.2093504</td>
</tr>
</tbody>
</table>

Table 10. Approximate solution of $x(t)$ for $\alpha = 0.75$

<table>
<thead>
<tr>
<th>$t$</th>
<th>FDTM</th>
<th>ADM</th>
<th>VIM</th>
<th>HAM</th>
<th>TF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.8492996</td>
<td>0.8492996</td>
<td>0.8492996</td>
<td>0.8492996</td>
<td>0.8492995</td>
</tr>
<tr>
<td>0.1</td>
<td>0.8016697</td>
<td>0.8016697</td>
<td>0.8016697</td>
<td>0.8016697</td>
<td>0.8016696</td>
</tr>
<tr>
<td>0.2</td>
<td>0.7978999</td>
<td>0.7978998</td>
<td>0.7978998</td>
<td>0.7978998</td>
<td>0.7979000</td>
</tr>
<tr>
<td>0.3</td>
<td>0.8250871</td>
<td>0.8250871</td>
<td>0.8250871</td>
<td>0.8250871</td>
<td>0.8250873</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8760144</td>
<td>0.8760143</td>
<td>0.8760143</td>
<td>0.8760143</td>
<td>0.8760146</td>
</tr>
<tr>
<td>0.5</td>
<td>0.9454582</td>
<td>0.9454581</td>
<td>0.9454581</td>
<td>0.9454581</td>
<td>0.9454582</td>
</tr>
<tr>
<td>0.6</td>
<td>1.0290757</td>
<td>1.0290756</td>
<td>1.0290756</td>
<td>1.0290756</td>
<td>1.0290757</td>
</tr>
<tr>
<td>0.7</td>
<td>1.1229595</td>
<td>1.1229594</td>
<td>1.1229594</td>
<td>1.1229594</td>
<td>1.1229592</td>
</tr>
<tr>
<td>0.8</td>
<td>1.2234368</td>
<td>1.2234365</td>
<td>1.2234365</td>
<td>1.2234365</td>
<td>1.2234363</td>
</tr>
<tr>
<td>0.9</td>
<td>1.3269767</td>
<td>1.3269759</td>
<td>1.3269759</td>
<td>1.3269759</td>
<td>1.3269757</td>
</tr>
</tbody>
</table>

The approximate solution of $y(t)$ and $z(t)$ for $\alpha = 1$ are obtained by solving \ref{eq:68} with step size of 0.003 and compared with analytical solutions in figure 2. Figure 3 shows TF solutions for different values of $\alpha$. 
Table 11. Approximate solution of $x(t)$ for $\alpha = 0.5$

<table>
<thead>
<tr>
<th>$t$</th>
<th>FDTM</th>
<th>ADM</th>
<th>VIM</th>
<th>HAM</th>
<th>TF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7642925</td>
<td>0.7642924</td>
<td>0.7642924</td>
<td>0.7642925</td>
<td>0.7642694</td>
</tr>
<tr>
<td>0.1</td>
<td>0.7545096</td>
<td>0.7545096</td>
<td>0.7545096</td>
<td>0.7545097</td>
<td>0.7544962</td>
</tr>
<tr>
<td>0.2</td>
<td>0.7903162</td>
<td>0.7903161</td>
<td>0.7903161</td>
<td>0.7903162</td>
<td>0.7903066</td>
</tr>
<tr>
<td>0.3</td>
<td>0.8524950</td>
<td>0.8524950</td>
<td>0.8524950</td>
<td>0.8524950</td>
<td>0.8524876</td>
</tr>
<tr>
<td>0.4</td>
<td>0.9323247</td>
<td>0.9323246</td>
<td>0.9323246</td>
<td>0.9323247</td>
<td>0.9323186</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0242052</td>
<td>1.0242051</td>
<td>1.0242051</td>
<td>1.0242052</td>
<td>1.0242000</td>
</tr>
<tr>
<td>0.6</td>
<td>1.1237906</td>
<td>1.1237905</td>
<td>1.1237905</td>
<td>1.1237906</td>
<td>1.1237861</td>
</tr>
<tr>
<td>0.7</td>
<td>1.2273291</td>
<td>1.2273291</td>
<td>1.2273291</td>
<td>1.2273291</td>
<td>1.2273251</td>
</tr>
<tr>
<td>0.8</td>
<td>1.3313915</td>
<td>1.3313915</td>
<td>1.3313915</td>
<td>1.3313916</td>
<td>1.3313879</td>
</tr>
<tr>
<td>0.9</td>
<td>1.4327552</td>
<td>1.4327552</td>
<td>1.4327552</td>
<td>1.4327552</td>
<td>1.4327519</td>
</tr>
</tbody>
</table>

Table 12. Approximate solution of $x(t)$ for $\alpha = 1$

<table>
<thead>
<tr>
<th>$t$</th>
<th>FDTM</th>
<th>ADM</th>
<th>VIM</th>
<th>HAM</th>
<th>TF</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.9048374</td>
<td>0.9048374</td>
<td>0.9048374</td>
<td>0.9048374</td>
<td>0.9048374</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.8187308</td>
<td>0.8187307</td>
<td>0.8187307</td>
<td>0.8187308</td>
<td>0.8187308</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.7408182</td>
<td>0.7408182</td>
<td>0.7408182</td>
<td>0.7408182</td>
<td>0.7408182</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.6703201</td>
<td>0.6703200</td>
<td>0.6703200</td>
<td>0.6703201</td>
<td>0.6703199</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.6065307</td>
<td>0.6065306</td>
<td>0.6065306</td>
<td>0.6065307</td>
<td>0.6065306</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.5488116</td>
<td>0.5488114</td>
<td>0.5488114</td>
<td>0.5488115</td>
<td>0.5488115</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.4965853</td>
<td>0.4965853</td>
<td>0.4965853</td>
<td>0.4965853</td>
<td>0.4965853</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>0.4493290</td>
<td>0.4493289</td>
<td>0.4493289</td>
<td>0.449329</td>
<td>0.4493288</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.4065697</td>
<td>0.4065696</td>
<td>0.4065696</td>
<td>0.4065697</td>
<td>0.4065697</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>0.3678795</td>
<td>0.3678794</td>
<td>0.3678794</td>
<td>0.3678794</td>
<td>0.3678794</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.3678795</td>
<td>0.3678794</td>
<td>0.3678794</td>
<td>0.3678794</td>
<td>0.3678794</td>
<td></td>
</tr>
</tbody>
</table>

Example 4
Consider the following linear fractional order differential-algebraic equation

\[
\begin{align*}
D_\alpha^* x(t) - ty'(t) + x(t) - (t + 1) y(t) &= 0, \\
y(t) - \sin t &= 0, \quad t \in [0, 1], 0 < \alpha \leq 1
\end{align*}
\] (69)

with initial conditions $x(0) = 1$, $y(0) = 0$.

For $\alpha = 1$, the given problem has exact solution; $x(t) = e^{-t} + t \sin t$ and $y(t) = \sin t$.

**Solution:**

Equation (69) can be reduced to the following form.

\[
D_\alpha^* x(t) = t \cos t + (1 + t) \sin t - x(t)
\] (70)

Performing fractional integration on both sides of above equation,

\[
x(t) = x(0) + J^\alpha (f_1(t)) - J^\alpha (x(t))
\] (71)
where $f_1(t) = t \cos t + (1 + t) \sin t$.

Approximating the functions: $x(0)$, $f_1(t)$ and $x(t)$ in the TF domain,

$$
\begin{align*}
  x(0) & \approx C^T_0 T_1(t) + D^T_0 T_2(t) \\
  f_1(t) & \approx C^T_0 T_1(t) + D^T_0 T_2(t) \\
  x(t) & \approx C^T_1 T_1(t) + D^T_1 T_2(t)
\end{align*}
$$

(72)

From equations 41, 71 and 72, we get the following expressions for $C^T_1$ and $D^T_1$.

$$
\begin{align*}
  C^T_1 &= C_1 + (C_2 P_{\alpha_1} + D_2 P_{\alpha_2}) - (C_1 P_{\alpha_1} + D_1 P_{\alpha_2}) \\
  D^T_1 &= D_1 + (C_2 P_{\alpha_2} + D_2 P_{\alpha_4}) - (C_1 P_{\alpha_2} + D_1 P_{\alpha_4})
\end{align*}
$$

(73)

As shown in tables 9 to 11, the TF solutions are in good agreement with the solutions obtained by FDTM [13], ADM [12], VIM [12] and HAM [13].

Table 13. Approximate solution of $x(t)$ for $\alpha = 0.75$

<table>
<thead>
<tr>
<th>$t$</th>
<th>FDTM</th>
<th>ADM</th>
<th>VIM</th>
<th>HAM</th>
<th>TF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.8373931</td>
<td>0.8373929</td>
<td>0.837393</td>
<td>0.8373931</td>
<td>0.8373884</td>
</tr>
<tr>
<td>0.2</td>
<td>0.7494391</td>
<td>0.7494390</td>
<td>0.7494389</td>
<td>0.7494391</td>
<td>0.7494360</td>
</tr>
<tr>
<td>0.3</td>
<td>0.6816129</td>
<td>0.6816128</td>
<td>0.6816129</td>
<td>0.6816129</td>
<td>0.6816107</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6250322</td>
<td>0.6250321</td>
<td>0.6250322</td>
<td>0.6250322</td>
<td>0.6250306</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5760122</td>
<td>0.5760121</td>
<td>0.5760121</td>
<td>0.5760121</td>
<td>0.5760109</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5326238</td>
<td>0.5326236</td>
<td>0.5326237</td>
<td>0.5326238</td>
<td>0.5326228</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4937128</td>
<td>0.4937126</td>
<td>0.4937126</td>
<td>0.4937126</td>
<td>0.4937119</td>
</tr>
<tr>
<td>0.8</td>
<td>0.4585198</td>
<td>0.4585196</td>
<td>0.4585196</td>
<td>0.4585197</td>
<td>0.4585190</td>
</tr>
<tr>
<td>0.9</td>
<td>0.4265076</td>
<td>0.4265076</td>
<td>0.4265076</td>
<td>0.4265076</td>
<td>0.4265070</td>
</tr>
<tr>
<td>1</td>
<td>0.3972738</td>
<td>0.3972732</td>
<td>0.3972735</td>
<td>0.3972736</td>
<td>0.3972731</td>
</tr>
</tbody>
</table>

Table 14. Approximate solution of $x(t)$ for $\alpha = 0.5$

<table>
<thead>
<tr>
<th>$t$</th>
<th>FDTM</th>
<th>ADM</th>
<th>VIM</th>
<th>HAM</th>
<th>TF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.760891</td>
<td>0.7608910</td>
<td>0.760891</td>
<td>0.760891</td>
<td>0.7608617</td>
</tr>
<tr>
<td>0.2</td>
<td>0.6909262</td>
<td>0.6909261</td>
<td>0.6909260</td>
<td>0.6909262</td>
<td>0.6909113</td>
</tr>
<tr>
<td>0.3</td>
<td>0.6396502</td>
<td>0.6396501</td>
<td>0.6396501</td>
<td>0.6396505</td>
<td>0.6396406</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5970877</td>
<td>0.5970876</td>
<td>0.5970877</td>
<td>0.5970878</td>
<td>0.5970808</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5599926</td>
<td>0.5599927</td>
<td>0.5599928</td>
<td>0.55999926</td>
<td>0.5599872</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5268894</td>
<td>0.5268890</td>
<td>0.5268894</td>
<td>0.5268894</td>
<td>0.5268850</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4969694</td>
<td>0.4969651</td>
<td>0.4969653</td>
<td>0.4969640</td>
<td>0.4969604</td>
</tr>
<tr>
<td>0.8</td>
<td>0.4697024</td>
<td>0.4697065</td>
<td>0.4697067</td>
<td>0.4697022</td>
<td>0.4696991</td>
</tr>
<tr>
<td>0.9</td>
<td>0.4447444</td>
<td>0.4447740</td>
<td>0.4447576</td>
<td>0.4447448</td>
<td>0.4447420</td>
</tr>
<tr>
<td>1</td>
<td>0.4218206</td>
<td>0.4219688</td>
<td>0.4218510</td>
<td>0.4218207</td>
<td>0.4218184</td>
</tr>
</tbody>
</table>

Example 5

The linear fractional order differential-algebraic equation

$$
\begin{align*}
  \begin{cases}
    x(t) + y(t) &= e^{-t} + \sin t, & t \in [0, 1], \ 0 < \alpha \leq 1 \\
    D^\alpha_x x(t) + x(t) - y(t) + \sin t &= 0
  \end{cases}
\end{align*}
$$

(74)
Table 15. Computational times for Examples 2-5

<table>
<thead>
<tr>
<th>Example</th>
<th>$h$</th>
<th>CPU time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.002</td>
<td>117.516</td>
</tr>
<tr>
<td>3</td>
<td>0.003</td>
<td>114.130</td>
</tr>
<tr>
<td>4</td>
<td>0.002</td>
<td>15.802</td>
</tr>
<tr>
<td>5</td>
<td>0.002</td>
<td>13.182</td>
</tr>
</tbody>
</table>

subject to initial conditions $x(0) = 1, y(0) = 0$.

For the special case of $\alpha = 1$, we have analytical solution; $x(t) = e^{-t}$ and $y(t) = \sin t$.

**Solution:**

Equation 74 can be written as

$$x(t) = x(0) + J_\alpha (e^{-t}) - J_\alpha (2x(t))$$

(75)

In the TF domain,

$$x(0) \approx C_1^T T_1 m(t) + D_1^T T_2 m(t)$$
$$e^{-t} \approx C_2^T T_1 m(t) + D_2^T T_2 m(t)$$
$$x(t) \approx C_1^T T_1 m(t) + D_1^T T_2 m(t)$$

(76)

From equations 75 and 76 we get the following expressions for $C_1^T$ and $D_1^T$.

$$C_1^T = C_{10} + (C_{20} P_2^\alpha + D_{20} P_2^\alpha) - 2 \ast (C_{1} P_1^\alpha + D_{1} P_1^\alpha)$$
$$D_1^T = D_{10} + (C_{20} P_2^\alpha + D_{20} P_2^\alpha) - 2 \ast (C_{1} P_1^\alpha + D_{1} P_1^\alpha)$$

(77)

The approximate numerical solution of $x(t)$ for different values of $\alpha$ are obtained by solving equation 77 and compared with the solutions by FDTM [13], ADM [12], VIM [12] and HAM [13] in tables 12 to 14.

8. Conclusions

Present work proposes the application of orthogonal triangular function for solving differential algebraic equations (DAEs); the fractional DAEs in specific. The proposed generalized TF operational matrices are successfully implemented on set of test problems consisting of linear and non-linear fractional order differential-algebraic equations. The implementation of the TF estimate for Riemann-Liouville fractional order integral appears to be complicated and time consuming. However the solutions in examples 1 to 5 (tables 2 and 15) ultimately utilizes algebraic equation solvers and do not require much CPU time to yield approximate numerical solutions with good accuracy. The proposed generalized TF operational matrices can be deployed for the analysis of real process dynamics; including non-linear one consisting of fractional DAEs.

References


52 S.K. DAMARLA, M. KUNDU JFCA-2015/6(2)


Seshu Kumar Damarla
DEPARTMENT OF CHEMICAL ENGINEERING, NATIONAL INSTITUTE OF TECHNOLOGY ROURKELA, INDIA-769008
E-mail address: 511ch113@nitrkl.ac.in, seshu.chemical@gmail.com

Madhusree Kundu
DEPARTMENT OF CHEMICAL ENGINEERING, NATIONAL INSTITUTE OF TECHNOLOGY ROURKELA, INDIA-769008
E-mail address: mkundu@nitrkl.ac.in