Disturbance Attenuation Problem for Overactuated Systems with Actuator Saturation: A Control Allocation Based Approach

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Abstract— **In this paper, a two-stage control allocation based covariance controller design method is proposed to ensure a certain level of disturbance attenuation property for a class of overactuated systems. The proposed method is equally efficient as existing single-stage design technique for assigning a state or output covariance matrix. A set of sufficient conditions has been derived for which the actuator displacements in single-stage technique and in the proposed allocation based method become same. The design steps are formulated in the linear matrix inequality framework. In the presence of actuator saturation, the allocation based method yields better closed-loop performance compared to the single-stage design technique until the level of actuator saturation remains under the attainable range of the allocator. To elucidate the effectiveness of the proposed method, two flight control design examples have been presented.**

Index Terms—**Covariance control, control allocation, disturbance attenuation, overactuated systems.**

I. Introduction

Many safety-critical systems use redundant actuators to improve system performance, reliability and reconfigurability. To handle system redundancy efficiently, quite often a two-stage control design method is adopted. In the first stage, a controller is designed to generate a virtual command and then the command is redistributed among healthy actuators by an allocator. This method provides an effective platform for handling actuator saturation and faulty situation. In allocator based design, input matrix of a given state-space model is factorized into two full rank matrices [1], [2]. The right one is known as control effectiveness matrix and the left one is called the input distribution matrix. The control effectiveness matrix is used to design control allocator, whereas the latter one is considered as input matrix for controller design. On the contrary, in single-stage design technique, the actuator command is generated by controller to meet closed-loop objectives. This single-stage method is hence not capable of utilizing the redundancy of the system as is done in two-stage allocation based technique to tackle actuator saturation and faulty situation. In this paper, we explore two-stage control allocation based technique for disturbance attenuation problem for a class of overactuated systems, where it is shown that the proposed control allocation based method is equally efficient as single-stage technique to assign covariance matrix but adds extra advantages while the actuators get saturated.

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In last few decades, the disturbance attenuation problem has drawn much interest in different areas of control engineering, e.g., in antenna movement [3], path following objective of air vehicles [4], [5], systems with wind gust disturbance [3], etc. In this work, we consider a class of overactuated systems whose input state-space matrix can be factorized into two full rank matrices. This type of systems are found in aerospace applications [6], robotics, underwater and ground vehicles [7], [8], [2], etc. The main results of this paper are built on the concepts furnished in [9] and [10]. It is shown that the closed-loop state covariance or output covariance is a measure of disturbance attenuation of a designed closed-loop system. This concept has widely been used in a variety of control applications ([11], [12], [13] and references therein). The covariance control provides a unified framework for handling many other control and estimation problems [10]. It is further worth noting that the closed-loop state covariance matrix is related to the closed-loop controllability gramian [9], [14], [10].

In this work, a sequence of design steps for two-stage control allocation based method has been proposed for a class of overactuated systems to achieve certain level of disturbance attenuation property of the closed-loop system. Disturbance considered is of wind gust type. Some computational complexities of the design have been circumvented by imposing a bound on the covariance matrix instead of assigning an exact one. The design is carried out in the linear matrix inequality framework. A set of sufficient conditions has also been derived to show that under a given condition, the actuator displacements in two design methods (i.e., the single-stage method and the proposed allocation based method) are identical. The effectiveness of the design approaches are demonstrated and compared by solving two flight control examples. It is also shown that, in the presence of actuator saturation, the closed-loop response is better in allocator based approach compared to the method of [10].

Notation

Notation is standard. M^T denotes the transpose of M. Let $M^{\perp}$ denote the matrix whose columns are the basis vectors of the null space of M. Let $\mathbb{R}^{n\times m}$ denote the set of real matrices with dimension $(n \times m)$. $\mathbb{R}^n$ represents the set of n-dimensional real vectors. $\|\cdot\|$ denotes the 2-norm. M^+ represents the Moore-Penrose pseudo inverse of M. I represents the identity matrix. The state covariance $X := \lim_{t\to\infty} \mathcal{E}\left[\bar{x}(t)\bar{x}(t)^T\right]$, where $\mathcal{E}\left[\cdot\right]$ represents the expectation.

II. Preliminaries

This section presents some terminologies and preliminary results which will streamline the main results of this paper.

Consider a stable linear time invariant (LTI) closed-loop system

$$\dot{\bar{x}} = A_c\bar{x} + E_c\bar{w},\ \bar{y} = C_c\bar{x} \tag{1}$$

where $\bar{x} \in \mathbb{R}^n$ is the state vector, $\bar{w} \in \mathbb{R}^l$ is the exogenous input vector and the output vector is $\bar{y} \in \mathbb{R}^m$. The controllability gramian X_g of the above system is the solution to the following Lyapunov equation [15]

$$A_cX_g + X_gA_c^T + E_cE_c^T = 0. \tag{2}$$

Considering $E_cE_c^T \geq 0$ and using the Lyapunov lemma [15], we have $X_g = X_g^T > 0$ iff $(A_c,\ E_c)$ is controllable. If the exogenous signal covariance is W, the state covariance matrix X is the solution to the following equation [9], [16]

$$A_cX + XA_c^T + E_cWE_c^T = 0. \tag{3}$$

This is worth noting that $X = X_g$ when $W = I$. Thus an assigned matrix X depends on the closed-loop system matrix A_c that contains controller parameters to be designed. Eq. (3) or, Eq. (2) with $W = I$ plays an important role in assigning state covariance, depicted later in Theorems 1 and 2, where controller design methods are given to assign a state covariance for the closed-loop system.

The following lemma is instrumental for controller synthesis to assign state covariance for a closed-loop system.

***Lemma** 1 ([10]):* Given a symmetric matrix Ω and an LTI continuous time system as in Eq. (1) where $\bar{w}$ is the stochastic white process noise with covariance I, then the following statements are equivalent:
(i) The system is asymptotically stable, and the output covariance is bounded above by Ω, i.e., $\lim_{t\to\infty} \mathcal{E}\left[\bar{y}(t)\bar{y}(t)^T\right] < \Omega$.
(ii) There exists a matrix $X > 0$ such that $C_cXC_c^T < \Omega,\ A_cX + XA_c^T + E_cE_c^T < 0.$ ◇

Note that the system considered in Eq. (1) is strictly proper. When $C_c = I$, Ω in Lemma 1 becomes an upper bound for both the state and output covariance matrices. The second condition of Lemma 1 is key to controller synthesis.

Next we present the Finsler's lemma [17], [18] which will be required to interpret the results of Theorem 2.

***Lemma** 2 ([17], [18]):* Let matrices $\mathfrak{B} \in \mathbb{R}^{n\times m}$ and $\mathcal{Q} \in \mathbb{R}^{n\times n}$ be given. Suppose rank$(\mathfrak{B}) < n$ and $\mathcal{Q} = \mathcal{Q}^T$. Let $(\mathfrak{B}_r,\ \mathfrak{B}_l)$ be any two full rank factors of $\mathfrak{B}$ so that $\mathfrak{B} = \mathfrak{B}_l\mathfrak{B}_r$, and define $\mathfrak{D} := \left(\mathfrak{B}_r\mathfrak{B}_r^T\right)^{-1/2} B_l^+$. Then $\mu\mathfrak{B}\mathfrak{B}^T - \mathcal{Q} > 0$ holds for some $\mu \in \mathbb{R}$ iff $P := \mathfrak{B}^\perp\mathcal{Q}\mathfrak{B}^{\perp T} < 0$ and μ is given by $\mu > \mu_{min} = \lambda_{max}\left[\mathfrak{D}\left(\mathcal{Q} - \mathcal{Q}\mathfrak{B}^\perp P^{-1}\mathfrak{B}^{\perp T}\mathcal{Q}\right)\mathfrak{D}^T\right].$ ◇

III. Main Results

In this section, a systematic design procedure is proposed for disturbance attenuation problem for a class of overactuated systems with actuator saturation, where the input state-space matrix is assumed to have the number of columns more than the number of state variables. It is worth noting that, with proper assumption, the input matrix for the above class of overactuated systems can be factorized into two full rank matrices [1] and by adopting control allocation technique, an effective closed-loop system can be designed in two steps. The control allocation based two-stage design method can assign the same state covariance matrix that can be assigned by the method of [10] and [14]. Thus, the proposed control allocation based method is an alternative technique for covariance control for overactuated systems. Further in this paper it is emphasized that the above two methods, i.e., the proposed one and the method of [10], are comparable in terms of the closed-loop performance, however, the control allocation based proposed method is advantageous while the actuators get saturated.

We present below a brief overview of the covariance control and control allocation methods, which will be required to establish main results of this section.

A. Covariance control

Let us consider the following system

$$\dot{\bar{x}} = A\bar{x} + B\bar{u} + E\bar{w},\ \bar{y} = C\bar{x} \tag{4}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{m\times n}$ and $E \in \mathbb{R}^{n\times l}$ are respectively the system matrix, the input matrix, the output matrix and the disturbance input matrix. We assume that the system is completely state controllable and all the states are available for feedback. We also assume that for closed-loop control a stabilizing state feedback control law $\bar{u} = G\bar{x}$ is applied. Then the following theorem provides all state covariance matrices that can be assigned by a set of static stabilizing state feedback controllers.

***Theorem** 1 ([10], [14]):* Let the state covariance $X \in \mathbb{R}^{n\times n}$ be a given positive definite matrix. Then the following statements are equivalent for the system given in Eq. (4).
(i) There exists a static stabilizing state feedback control gain G that assigns X for the closed-loop system.
(ii) X satisfies $(I - BB^+)\left(AX + XA^T + W\right)(I - BB^+) = 0$ and the set of controllers, which can assign state covariance X, is given by $G = -\frac{1}{2}B^+\left(AX + XA^T + W\right)(2I - BB^+)X^{-1} + B^+S_fBB^+X^{-1} + (2I - BB^+)Z_f$ where Z_f is arbitrary, S_f is an arbitrary skew-symmetric matrix and W is the noise covariance matrix. ◇

The proof of this theorem is given in [10]. The matrix equality condition in Theorem 1 is not straightforward to solve [10]. To circumvent this difficulty, the above result is extended, presented in Theorem 2 below, where a covariance matrix is assigned for a given upper bound with unity noise covariance. In this paper, this result is further explored to propose a control allocation based new covariance control technique for overactuated systems.

***Theorem** 2 ([10]):* Consider the system given in Eq. (4) and let a symmetric matrix Ω be given. Then the following statements are equivalent:

(i) There exists a static stabilizing state feedback gain G such that $\lim_{t\to\infty} \mathcal{E}\left[\bar{y}(t)\bar{y}(t)^T\right] < \Omega$.
(ii) There exists a matrix $X > 0$ such that $CXC^T < \Omega,\ \ B^\perp\left(AX + XA^T + EE\right)B^{\perp T} < 0.$
(iii) There exist a scalar $\gamma > 0$ and a matrix $Q > 0$ such that a positive definite solution $P > 0$ to $PA +$

$A^TP + P\left(\frac{1}{\gamma^2}EE^T - BB^T\right)P + Q = 0$ satisfying $\gamma^2 CP^{-1}C^T < \Omega$. In this case, all static stabilizing state feedback gains are given by $G = -B^TP + \mathfrak{L}Q^{1/2}$ where $\mathfrak{L}$ is an arbitrary matrix with $\|\mathfrak{L}\| < 1$. $\diamond$

Note that, the equivalence between the condition (i) and conditions (ii) and (iii) is established by applying Lemma 1. On the other hand, Lemma 2 shows the equivalence between the condition (ii) and condition (iii). From condition (iii), we have $P = P^T > 0$ and $\gamma > 0$ such that $\exists\ Q = Q^T > 0$ satisfying $A\left(\gamma^2P^{-1}\right) + \left(\gamma^2P^{-1}\right)A^T + EE^T - \gamma^2BB^T = -\gamma^2P^{-1}QP^{-1} \iff \exists\ P = P^T > 0$ and $\gamma > 0$ such that $A\left(\gamma^2P^{-1}\right) + \left(\gamma^2P^{-1}\right)A^T + EE^T - \gamma^2BB^T < 0$.

Defining $X = \left(\gamma^2P^{-1}\right)$ and $\beta = \gamma^2$, the above inequality becomes

$$AX + XA^T + EE^T - \beta BB^T < 0. \tag{5}$$

Now applying Lemma 2, it is apparent that inequality (5) along with the constraint $C(\gamma^2\ P^{-1})C^T < \Omega$ hold, if and only if, the condition (ii) of Theorem 2 is satisfied. Also note that the inequality (5) is numerically tractable as it is in the form of LMI constraint to find X and β in single step to design a static stabilizing state feedback controller for assigning a covariance matrix.

B. Allocator

We consider in this paper a sub-class of overactuated systems whose input state-space matrix can be factorized into two full rank matrices. A basic feedback control structure for an overactuated system with allocator is shown in Fig. 1, where $\bar{r} \in \mathbb{R}^m$ is the reference command, $\bar{v} \in \mathbb{R}^m$ is the virtual control input generated by the controller, $\bar{u} \in \mathbb{R}^p$ is the actuator command generated by the allocator and $\bar{y} \in \mathbb{R}^m$ is the output vector of the system. Let an LTI overactuated system be given as

$$\dot{\bar{x}} = A\bar{x} + B_v\bar{v},\ \ \bar{y} = C\bar{x},\ \ B_u\bar{u} = \bar{v}. \tag{6}$$

where $A \in \mathbb{R}^{n\times n}$ is the system matrix, $B \in \mathbb{R}^{n\times p}$ is the input matrix, $C \in \mathbb{R}^{m\times n}$ is the output matrix and $p > m$. We assume that the pair $(A,\,B)$ is controllable, and the input matrix B can be factorized into two full rank matrices as $B = B_vB_u$, where B_v has full column rank and B_u has full row rank. After factorization, a stabilizing controller is designed considering the input matrix as B_v. With the help of the allocator, the output of the controller (i.e., the virtual command) is then distributed among actuators maintaining all physical limits.

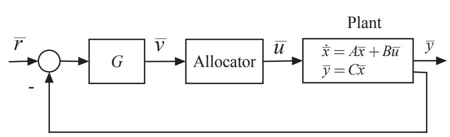

Fig. 1: A schematic block diagram for an overactuated system with allocator

An allocator solves the following optimization problem.

$$\min_{\bar{u}} \|\bar{u}^T\bar{u}\| \quad \text{subject to } B_u\bar{u} = \bar{v}, \text{ and } \bar{u}_{min} \le \bar{u} \le \bar{u}_{max}, \tag{7}$$

where B_u is the control effectiveness matrix and $\bar{u}_{min/max}$ are actuator limits. If the condition $B_u\bar{u} = \bar{v}$ holds, the open-loop system dynamics is governed by Eq. (6). Note that the full rank factorization $B = B_vB_u$ ensures the complete state controllability of the pair $(A,\,B_v)$. Different numerical algorithms to solve Eq. (7) are available in [2], [19]. In the following subsection, systematic design procedures are presented for disturbance attenuation problem using the control allocation based approach and the method of [10].

C. Covariance controller design for overactuated systems

Using the conditions of Theorem 2, new design methods are proposed for the overactuated system given in Eq. (4).

Suppose the state-space matrices A, B, C, E and a bound for covariance matrix Ω are given. Suppose further that (A, B) is controllable and $B = B_vB_u$ is a full rank factorization.

Design 1 *(using the method of [10]):*

Step 1: Find $\beta > 0$ and $X > 0$ by solving the feasible problem with following LMI constraints

$$\beta BB^T - AX - XA^T - EE^T > 0,\ CXC^T < \Omega. \tag{8}$$

Step 2: Calculate $P = \beta X^{-1}$ and $Q = \beta^{-1}P\left(\beta BB^T - AX - XA^T - EE^T\right)P$.

Step 3: Choose a matrix $\mathfrak{L}$ such that $\|\mathfrak{L}\| < 1$.

Step 4: $G_1 = -B^TP + \mathfrak{L}Q^{1/2}$.

Step 5: The actuator command is $\bar{u}_1 = G_1\bar{x}$.

Design 2 *(using control allocation method):*

Step 1 Factorize $B = B_vB_u$, where B_v has full column rank and B_u has full row rank.

Steps 2–4: Follow Step 1 to Step 3 of Design 1 replacing B with B_v.

Step 5: $G_2 = -B_v^TP + \mathfrak{L}Q^{1/2}$.

Step 6: $\bar{v}_2 = G_2\bar{x}$ and $\bar{v}_2 = B_u\bar{u}_2$, where $\bar{v}_2$ and $\bar{u}_2$ are allocator and actuator commands respectively. $\diamond$

We now compare the effectiveness of Design 1 and Design 2 for covariance assignment. It will be shown in this section that both the design methods can assign same covariance matrix. However, the allocation based method is more efficient than the other to tackle actuator saturation and faulty situation. It is worth noting that in Design 2, a stabilizing controller is designed by considering B_v as input matrix while Design 1 considers the input matrix B for controller design. We now propose the main results of the paper.

Proposition 1: Consider the system given in Eq. (4). Suppose (A, B) is controllable and $B = B_vB_u$ is a full rank factorization. Then there exist static stabilizing state feedback controllers G_1 and G_2, which can assign same closed-loop state and output covariances via Design 1 and Design 2, respectively.

Proof: First we prove that assigned state covariances are same for both the design methods. If B_v has full column rank and B_u has full row rank, then the pseudo inverse $(B_vB_u)^+ = B_u^+B_v^+$, and for a full row rank matrix B_u, $B_u^+ = B_u^T(B_uB_u^T)^{-1}$ and $B_uB_u^+ = I$. Using these relationships, from Theorem 1 we have $(I - BB^+) = (I - B_vB_v^+)$

for both the methods. Therefore from Theorem 1, assignable state covariances are same, and consequently the output covariances are also equal as C is identical for both the methods.

On the other hand, since B_u has full row rank, the basis vectors of left null space of B_v and B are same, i.e., $B_v^{\perp} = B^{\perp}$. Now using condition (ii) of Theorem 2, state covariances obtained are same for both the methods since they satisfy the same set of LMI constraints. The output covariances are also equal as C is identical for both the methods. Hence the proof is completed. □

Remark: Considering Design 1 and Design 2, it is apparent from Proposition 1 that the assigned state covariance matrix X becomes same in two methods but the values of $\gamma = \sqrt{\beta}$ are different. From Theorem 2, it is worth noting that the upper bound for covariance is independent of the choice of matrix $\mathfrak{L}$ and hence, one can choose $\mathfrak{L}$ as null matrix but it affects the performance of the closed-loop system.

Note that, although the disturbance rejection properties are same in both the approaches, the actuator displacement (assumed to be same as actuator command) may not be same. To ensure this, designed controller needs to be retuned by setting a new value to the free variables γ and $\mathfrak{L}$, presented next in Proposition 2. Now if the control inputs get saturated due to actuator constraints, the control allocation based covariance control method (Design 2) avoids actuator saturation by redistributing attainable control command among healthy actuators and thereby the closed-loop nominal performance can be retained; however, for the same level of control input, Design 1 loses closed-loop performance owing to the effect of saturation.

***Proposition** 2:* Consider the system given in Eq. (4). Suppose (A, B) is controllable, $B = B_v B_u$ is a full rank factorization, and actuator commands $\bar{u}_1$ and $\bar{u}_2$ are obtained from Design 1 and Design 2 respectively. Then the following statements hold.
(a) Suppose γ_2 and a static stabilizing state feedback controller G_2 are obtained by considering $\mathfrak{L}_2 = 0$ from Design 2 to assign a state covariance X. Then there exists a static stabilizing state feedback controller G_1 that generates actuator command $\bar{u}_1 = \bar{u}_2$ and assigns the same state covariance X if $\exists\ \beta_1 > 0$ such that the following LMI is satisfied:

$$\begin{bmatrix} -\beta_1 I & \left(B^T\beta_1 - B_u^+ B_v^T \gamma_2^2\right) \\ \left(B^T\beta_1 - B_u^+ B_v^T \gamma_2^2\right)^T & -\Phi_1 \end{bmatrix} < 0, \qquad (9)$$

where $\Phi_1 = \beta_1 B B^T - AX - XA^T - EE^T$.
(b) Suppose γ_1 and a static stabilizing state feedback controller G_1 are obtained by considering $\mathfrak{L}_1 = 0$ from Design 1 to assign a state covariance X. Then there exists a static stabilizing state feedback controller G_2 that produces actuator command $\bar{u}_2 = \bar{u}_1$ and assigns the same state covariance X if $\exists\ \beta_2 > 0$ such that the following LMI is satisfied:

$$\begin{bmatrix} -\beta_2 I & \left(B_v^T\beta_2 - (B_u B_u^T) B_v^T \gamma_1^2\right) \\ \left(B_v^T\beta_2 - (B_u B_u^T) B_v^T \gamma_1^2\right)^T & -\Phi_2 \end{bmatrix} < 0 \qquad (10)$$

where $\Phi_2 = \beta_2 B_v B_v^T - AX - XA^T - EE^T$. ◇

Proof: (a) In this proof, variables related to Design 1 and Design 2 are respectively denoted by the subscripts 1 and 2. Suppose Design 2 is followed with $\mathfrak{L}_2 = 0$ and we have a static stabilizing controller $G_2 = -B_v^T P_2$ and $\bar{v}_2 = -B_v^T P_2 \bar{x}$. Replacing $\bar{v}_2$ in the solution of the optimization problem (7), we get $\bar{u}_2 = B_u^+ \bar{v}_2 = -B_u^+ B_v^T P_2 \bar{x}$. Suppose another controller is obtained for the same state-space matrices by following the steps of Design 1 where an arbitrary $\mathfrak{L}_1$ is chosen with $\|\mathfrak{L}_1\| < 1$. Then we have $G_1 = \left(-B^T P_1 + \mathfrak{L}_1 Q_1^{1/2}\right)$ and $\bar{u}_1 = \left(-B^T P_1 + \mathfrak{L}_1 Q_1^{1/2}\right)\bar{x}$. Since $B = B_v B_u$ is a full rank factorization via supposition, from Proposition 1 it is inferred that the assigned disturbance covariance matrix X is same in both the cases. Now if $-B_u^+ B_v^T P_2 = -B^T P_1 + \mathfrak{L}_1 Q_1^{1/2}$ we have $\bar{u}_2 = \bar{u}_1$. From the equality condition we get $\mathfrak{L}_1 = \left(B^T P_1 - B_u^+ B_v^T P_2\right) Q_1^{-1/2}$. Now $\|\mathfrak{L}_1\| < 1 \iff \mathfrak{L}_1 \mathfrak{L}_1^T < I$. Defining $P_1 = \gamma_1^2 X^{-1}$, $P_2 = \gamma_2^2 X^{-1}$, $X^{-1} Q_1^{-1} X^{-1} = \left(\gamma_1^2 \Phi_1\right)^{-1}$, $\gamma_1^2 = \beta_1$, and using Schur complement lemma, the condition $\mathfrak{L}_1 \mathfrak{L}_1^T < I$ is equivalent to (9). Since (9) implies (8), by solving (9) as a feasible problem one gets a covariance matrix X that is same as in Design 2. Hence the proof of part (a) is completed.

(b) Suppose $G_1 = -B^T P_1$ is obtained from Design 1 considering $\mathfrak{L}_1 = 0$. Using the same state-space matrices $G_2 = \left(-B_v^T P_2 + \mathfrak{L}_2 Q_2^{1/2}\right)$ is another controller obtained from Design 2 where $\|\mathfrak{L}_2\| < 1$. Now if $B_u^+ G_2 = G_1$, we have $\bar{u}_2 = \bar{u}_1$. Since $B_u B_u^+ = I$, the condition $\|\mathfrak{L}_2\| < 1$ is equivalent to (10), obtained by defining $\gamma_2^2 = \beta_2$ and following the same line of proof given in part (a). Since (10) implies (8), solving (10) as a feasible problem one gets a covariance matrix X that is same as obtained in Design 1. Hence the proof of part (b) is completed. □

In part (a) of Proposition 2, G_2 is obtained using Design 2 with $\mathfrak{L}_2 = 0$. In order to impose additional constraint $\bar{u}_1 = \bar{u}_2$, Step 1 of Design 1 is modified by replacing the constraint (8) with (9). Then following Step 2, P_1 and Q_1 are calculated. From Step 4, G_1 is calculated with $\mathfrak{L}_1 = \left(B^T P_1 - B_u^+ B_v^T P_2\right) Q_1^{-1/2}$, where P_2 is the matrix obtained from Design 2.

In part (b) of Proposition 2, G_1 is obtained using Design 1 with $\mathfrak{L}_1 = 0$. To satisfy the constraints $\bar{u}_2 = \bar{u}_1$, Step 2 of Design 2 is modified by replacing the constraint (8) with (10). Then following Step 3, P_2 and Q_2 are calculated. From Step 5, G_2 is calculated with $\mathfrak{L}_2 = \left(B_v^T P_2 \ - \left(B_u B_u^T\right) B_v^T P_1\right) Q_2^{-1/2}$, where P_1 is obtained from Design 1.

Results developed in this section are applicable only in the linear zone of actuators. In the next subsection, we briefly explain how an allocator helps to operate the closed-loop system in the linear zone of actuators when some of them reach saturation limit.

D. Consideration of actuator saturation

A high control effort demands larger actuator displacement. But due to presence of position limits of the actuators, they may get saturated. The actuator saturation problem is explicitly taken care of by the allocator in a two-stage allocation based method. When the virtual command (from the controller) $\bar{v}$ is large, the actuator command, obtained as $\bar{u} = B_u^+\bar{v}$, may reach or cross position limit of some actuators. Then a new command $\bar{u}$ is generated to meet (6) by minimizing another objective function $\|B_u\bar{u}-\bar{v}\|$ along with the objective given in (7). As long as the virtual command is attainable, $\bar{v} = B_u\bar{u}$, i.e., the virtual command driving the system is unaltered, the closed-loop system is in linear zone of actuators. The nominal performance of the controller is retained. But actuator commands generated via Design 1 and Design 2 are not equal, i.e., $\bar{u}_1 \neq \bar{u}_2$. When virtual command is unattainable, the system operates outside the linear zone. So in the presence of an allocator the linear operating zone is effectively increased. In the next section, through examples, we show that an allocator helps a system to retain its closed-loop performance without significant alteration.

IV. Simulation Results

In this section, two numerical examples have been presented to demonstrate the effectiveness of the proposed method, especially the performance of the designed controllers while actuators get saturated. In the first example, ADMIRE model [1] is considered, and a Satellite Launch Vehicle model [20] is taken up in the second example. In simulation studies, a scalar wind gust disturbance is considered, and so the disturbance input matrix E is a single-column matrix. Here results are compared considering following three cases— Case 1: Plant model without actuator saturation and with controller G_1; Case 2: Plant model with actuator saturation and with controller G_2 where the allocation algorithm follows Eq. (7); Case 3: Plant model with actuator saturation and with controller G_1.

Example 1: The ADMIRE model describes a small single engine fighter with delta-canard configuration. The parameters of the model are for Mach 0.22, at altitude 3000 m. The linearized aircraft model is considered neglecting the actuator dynamics. In this work, the control surfaces are viewed as pure moment generators, and their influence on the rate of change of angle of attack and the rate of change of sideslip angle are neglected.

The approximate models for Designs 1 and 2 are given by Eq. (4). Here the components of $\bar{v}$ are the angular accelerations in roll, pitch and yaw produced by the control surfaces. The system matrices other than E are given in [1]. Output covariance values obtained for different disturbance input matrices (E) are listed in Table I. We assume the

TABLE I: Covariance bound obtained in ADMIRE system for Design 1 and Design 2

E^T	$\lambda_{max}(\Omega)$	$\lambda_{max}\left(CXC^T\right)$	
		Design 2	Design 1
$[0, 1, 0, 1, 0]$	25	20.40	20.40
$[1, 1, 1, 1, 0]$	25	20.55	20.55

angle of attack commands of $5°$ and $0.2°$ are applied at $t = 0$ s and $t = 2$ s, respectively. Sideslip angle is kept zero, and a roll rate command of $10°/s$ is applied at $t = 3$ s. It is shown in Figs. 2 and 3 that the actuator command, as well as the output, with two different designs are same. This result indicates that the relation established in Proposition 2 is correct. Further, a step disturbance of amplitude 2.5 (considering $E = [1,\ 1,\ 1,\ 1,\ 0]^T$) is applied at $t = 3.5$ s for the duration of 0.01 s, and it is observed that Actuator-2 reaches saturation. Due to the disturbance, it is shown in Fig. 2 that roll rate, for Design 1, deviates more from the response of the system without saturation. But in the presence of the allocator (in Design 2) the deviation is small. It is seen in Fig. 3 that Actuators-1, 3 and 4 are used by the allocator to nullify the saturation effect as much as possible.

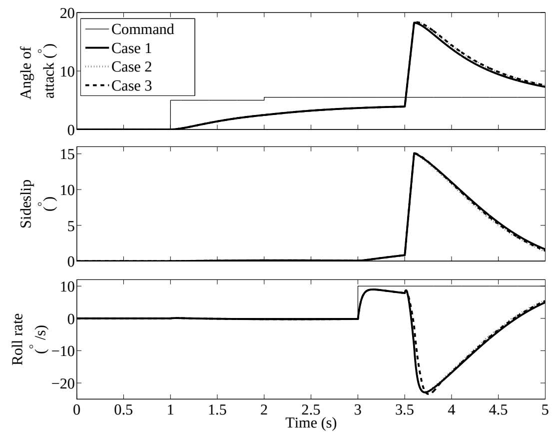

Fig. 2: Output response of closed-loop ADMIRE system models.

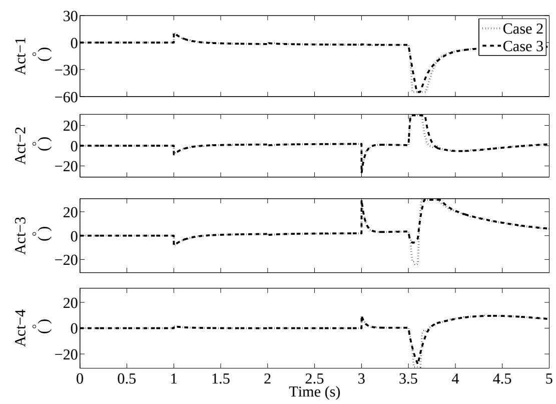

Fig. 3: Actuator responses of closed-loop ADMIRE system models.

Example 2 : A Satellite Launch Vehicle (SLV) model with eight actuators is considered in this example. Eight actuators control four thrusters. The linearized model neglecting actuator dynamics is described by Eq. (4). The system matrices except E are given in [20]. Output covariance values obtained for different disturbance input matrices (E) are listed in Table II. The pitch, yaw and roll command of $1.5°$, $1.5°$ and $0.5°$ are applied at $t = 0$ s, 0.02 s and 0.1 s, respectively. Since there is no saturation in actuators, output and actuator responses are same for both the designs up to $t = 2$ s as shown in Figs. 4 and 5. This result is as

TABLE II: Covariance bound obtained in SLV system for Design 1 and Design 2

E^T	$\lambda_{max}(\Omega)$	$\lambda_{max}\left(CXC^T\right)$	
		Design 2	Design 1
$[0, 1, 0, 1, 0, 1]$	25	14.58	14.58
$[1, 1, 1, 1, 1, 1]$	25	19.25	19.25

expected from Proposition 2. At $t = 2$ s, a disturbance of amplitude 0.3 is applied for the duration of 0.1 s. Assume $E = [1,\ 1,\ 1,\ 1,\ 1,\ 1]^T$. When the disturbance appears, Actuators-1 and 3 get saturated. Now the system with allocator based method nullify the effect as much as possible using rest of the actuators. As a result, the output response of the allocator based method (Design 2) is less affected compared to the output of the system obtained using Design 1. This is shown in Fig. 4.

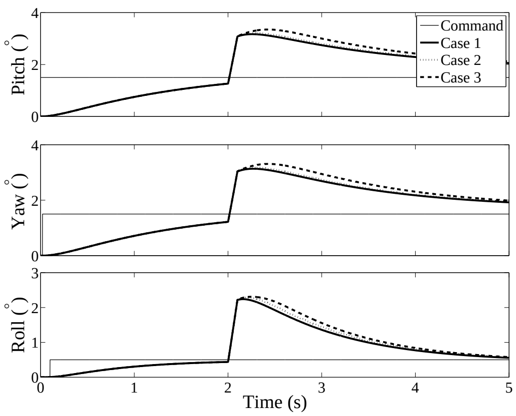

Fig. 4: Output response of closed-loop SLV system models.

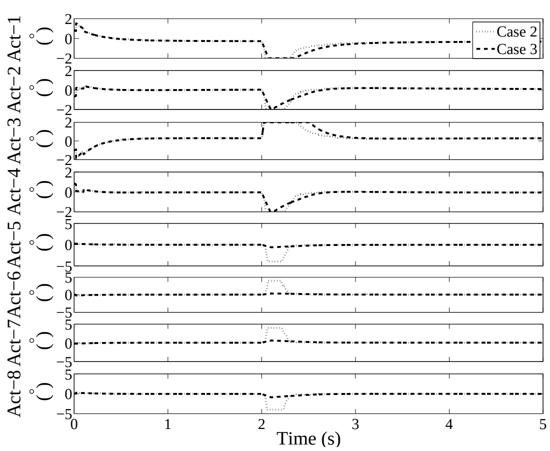

Fig. 5: Actuator response of closed-loop SLV system.

V. Conclusions

In this paper, a sequence of design steps is proposed for covariance control using the control allocation based technique for a class of overactuated systems. It has been shown that the allocator based method (Design 2) can assign same covariance matrix as obtained by the existing methods of [10] and [14]. The allocator based method (Design 2) is successfully applied to two flight control examples. Conditions are also derived for which the same control command can be generated by two methods, i.e., the allocator based method (Design 2) and the method of [10] (Design 1). Simulation results show that in the presence of actuator saturation, Design 2 outperforms Design 1.

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