Indian J. pure appl. Math., 23(10): 693-696, October 1992

A GENERALIZED NONLINEAR COMPLEMENTARITY PROBLEM OF MATHEMATICAL PROGRAMMING IN BANACH SPACES

A. BEHERA AND G. K. PANDA

Department of Mathematics, Regional Engineering College
Rourkela 769 008 (*Orissa*)

(*Received* 1 *April* 1992; *accepted* 26 *May* 1992)

A generalized version of the existence theorem on nonlinear complementarity problem of mathematical programming in a reflexive real Banach space for arbitrary closed convex cone is proved. Furthermore, in the already known version of the existence theorem of the same problem, the solution is shown to be unique under different assumptions.

1. INTRODUCTION

Let X be a reflexive real Banach space and let X^* be its dual. Let the value of $f \in X^*$ at $x \in X$ be denoted by (f, x). Let C be a closed convex cone in X with the vertex at 0. The 'polar' of C is the cone C^* defined by

$$C^* = \{f \in X^* : (f, x) \geq 0 \text{ for each } x \in C\}.$$

A mapping $T : C \to X^*$ is said to be 'monotone' if $(Tx - Ty, x - y) \geq 0$ for all $x, y \in C$ and 'strictly monotone' if strict inequality holds whenever $x \neq y$. T is said to be 'coercive' on C if

$$\frac{(Tx, x)}{\|x\|} \to \infty \text{ as } \|x\| \to \infty, \text{ for } x \in C.$$

T is called 'hemicontinuous' on C if for all $x, y \in C$, the map $t \mapsto T(ty + (1 - t)x)$ of $[0, 1]$ to X^* is continuous, when X^* is endowed with the weak* topology[3].

We will use the following result of Browder[1] (see also Mosco[2]) to prove our results.

Proposition A — Let T be a monotone and hemicontinuous map of a closed convex set K in X, with $0 \in K$, into X^*, and if K is not bounded, let T be coercive on K. Then there is an $x_0 \in K$ such that

$$(Tx_0, y - x_0) \geq 0 \text{ for all } y \in K. \qquad \ldots(1)$$

The inequalities of the form (1) are called 'variational inequalities'[3].

Using the above Proposition A the following theorem on nonlinear complementarity problem of mathematical programming in a reflexive real Banach space for arbitrary closed convex cone is proved by Nanda[3].

Theorem B — Let $T : C \to X^*$ be hemicontinuous, monotone and coercive on C. Then there exists x_0 such that

$$x_0 \in C,\ Tx_0 \in C^* \text{ and } (Tx_0, x_0) = 0. \qquad \ldots(2)$$

First we prove a generalized version of Theorem B. Next we show that x_0 in (2) is unique if T is assumed to be strictly monotone instead of being monotone. Also we give an example to show that x_0 in (2) is not unique if T is not strictly monotone.

2. A Generalized Complementarity Problem

We generalize Theorem B in the following sense, that is, there exist x_0 and a nontrivial closed convex subcone $\tilde{C}$ of C such that $x_0 \in \tilde{C}$, $Tx_0 \in \tilde{C}^*$ and $(Tx_0, y) = 0$ for all y in $\tilde{C}$. Theorem B is a particular case of the theorem given below.

Theorem — Let $T : C \to X^*$ be hemicontinuous, monotone and coercive on C. Then there exist x_0 and a nontrivial closed convex subcone $\tilde{C}$ of C such that

$$x_0 \in \tilde{C},\ Tx_0 \in \tilde{C}^* \text{ and } (Tx_0, y) = 0 \text{ for all } y \in \tilde{C}.$$

Proof : If $C = \{0\}$ then the theorem becomes trivial. Since C is a closed convex cone in the Banach space X, there exists a maximal linearly independent set of vectors in C, say, $\{x_i : i \in L\}$ such that each $x \in C$ can be written as

$$x = \sum_{i \in L} a_i x_i,\ a_i \geq 0.$$

By Proposition A, there exists $x_0 \in C$ such that

$$(Tx_0, y - x_0) \geq 0 \text{ for all } y \in C. \qquad \ldots(3)$$

Let

$$x_0 = \sum_{i \in L} b_i x_i,\ b_i \geq 0.$$

If $x_0 = 0$, then $b_i = 0$ for each $i \in L$. If $x_0 \neq 0$, then there exists $b_i > 0$ for some $i \in L$. Let

$$L' = \left\{ i \in L : b_i > 0 \text{ in } x_0 = \sum_{i \in L} b_i x_i,\ x_0 \neq 0 \right\} \subset L.$$

Now for any $j \in L'$ taking $y = x_0 + b_j x_j \in C$, from (3) we get $(Tx_0, b_j x_j) \geq 0$ for $j \in L'$. Now taking

$$y = x_0 - b_j x_j = \sum_{k \in L - \{j\}} b_k x_k \in C$$

again from (3), we get $(Tx_0, b_j x_j) \leq 0$, for $j \in L'$. Therefore $(Tx_0, b_j x_j) = 0$ for all $j \in L'$. Since $j \in L'$ (i.e., $b_j > 0$) we get $(Tx_0, x_j) = 0$ for all $j \in L'$. Let

$$\tilde{C} = \left\{ \sum_{k \in L'} c_k x_k : c_k \geq 0 \right\} \subset C.$$

$\tilde{C}$ is a nontrivial closed convex subcone of C. If $y \in \tilde{C}$, then

$$y = \sum_{k \in L'} c_k x_k, \; c_k \geq 0$$

and

$$(Tx_0, y) = \sum_{k \in L'} c_k \, (Tx_0, x_k) = 0.$$

We note that $x_0 \in \tilde{C}$ for if $x_0 \neq 0$, then

$$x_0 = \sum_{i \in L} b_i x_i = \sum_{i \in L'} b_i x_i \in \tilde{C}$$

since $b_i = 0$ for $i \in L - L'$. Clearly $Tx_0 \in \tilde{C}^*$. □

3. Uniqueness

Now we prove that if $T : C \to X^*$ of Theorem B is strictly monotone instead of being monotone, then the solution x_0 of (2) of Theorem B is unique.

Proposition – Let $T : C \to X^*$ be hemicontinuous, strictly monotone and coercive on C. Then there exists a unique x_0 such that

$$x_0 \in C, \; Tx_0 \in C^* \text{ and } (Tx_0, x_0) = 0.$$

Proof : Suppose that y_0 also satisfies the condition of the above proposition. By Proposition A, $(Tx_0, y_0 - x_0) \geq 0$ and $(Ty_0, x_0 - y_0) \geq 0$; and these two imply $(Tx_0, y_0) \geq 0$ and $(Ty_0, x_0) \geq 0$. Thus $(Tx_0, y_0) + (Ty_0, x_0) \geq 0$. On the other hand since T is strictly monotone we have $(Tx_0 - Ty_0, x_0 - y_0) \geq 0$ (equality holds if $x_0 = y_0$) ; on simplifying this and using $(Tx_0, x_0) = 0$, $(Ty_0, y_0) = 0$, we get $(Tx_0, y_0) + (Ty_0, x_0) \leq 0$. Thus $(Tx_0, y_0) + (Ty_0, x_0) = 0$. Since each term is nonnegative, we must have $(Tx_0, y_0) = 0$ and $(Ty_0, x_0) = 0$ and hence $(Tx_0 - Ty_0, x_0 - y_0) = 0$. Since T is strictly monotone we have $x_0 - y_0 = 0$. □

4. Example

The following example shows that the solution x_0 as obtained in the above proposition is not unique if T is not strictly monotone.

Let $X = \mathbb{R}$ and $C = \{x \in X : x \geq 0\}$, so that $C = C^*$. Let $T : C \to \mathbb{R}$ be defined by

$$T(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1 \\ \dfrac{x-1}{x} & \text{if } x > 1. \end{cases}$$

T is clearly hemicontinuous, monotone and coercive. But any point of the interval [0, 1] is a solution of $(Tx_0, x_0) = 0$, e.g., $(T\,½, ½) = 0$ and $(T\,1, 1) = 0$.

References

1. F. E. Browder, *Bull. Am. Math. Soc.* **71** (1965), 780-85.
2. U. Mosco, *Adv. Math.* **3** (1969), 510-85.
3. S. Nanda, *Indian J. pure appl. Math.* **18** (3) (1987), 215-18.