

Affine Kac-Moody symmetric spaces associated with untwisted Kac-Moody algebras

Saudamini Nayak, S.S.Rout and K.C.Pati*

Department of Mathematics, National Institute of Technology, Rourkela, Odisha-769 008 India.

In this paper we have computed all the affine Kac-Moody symmetric spaces which are tame Fréchet manifolds starting from the Vogan diagrams related to the affine untwisted Kac-Moody algebras. The detail computation of affine Kac-Moody symmetric spaces associated with $A_1^{(1)}$ and $A_2^{(1)}$ are shown algebraically to corroborate our method.

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I. INTRODUCTION

Finite dimensional symmetric spaces [20, 25] are rather well understood mathematical objects which have recently gained much importance in both mathematics and physics due to their intimate connections with random matrix theories, Riemannian geometries and their applications to many integrable systems, quantum transport phenomena (disordered system etc) [4, 5, 9, 22]. A compact irreducible symmetric space is either a compact simple Lie group G or a quotient G/K of a compact simple Lie group by the fixed point set of an involution ρ (or an open subgroup of it) and $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ is the decomposition of the Lie algebra \mathfrak{g} of group G into $+1$ and -1 eigenvalue spaces of ρ then K acts on \mathfrak{g} by adjoint representation leaving the decomposition invariant. The restriction of this action to \mathfrak{p} can be identified with the isotropy representation of G/K and we know that the isotropy representation of a symmetric space is polar.

With the advent of Kac-Moody algebras [15, 16, 21] which can be considered as the generalization of finite dimensional Lie algebras, naturally a search of infinite dimensional version of symmetric spaces began, the closest generalization being the affine Kac-Moody symmetric spaces. An affine Kac-Moody symmetric space is by definition either an affine Kac-Moody group \hat{G} (group type) or a quotient $\hat{G}/\hat{G}^{\hat{\rho}}$ of \hat{G} by the fixed point set of an involution of the second kind. In fact if $\hat{\mathfrak{g}} = \hat{\mathfrak{t}} + \hat{\mathfrak{p}}$ in the splitting of the Lie algebra of \hat{G} into the ± 1 eigen spaces of $\hat{\rho}$. Then the metric of \hat{G}/\hat{K} is the left invariant metric obtained from the restriction of the inner product of $\hat{\mathfrak{g}}$ to $\hat{\mathfrak{p}}$. In finite dimension, the isotropy representation is polar itself while in infinite dimension it leaves invariant a co-dimension-2 sub-manifold which can be identified with a (pre-)Hilbert space and the induced action on this space is a polar action by affine isometries.

The study on affine Kac-Moody symmetric spaces began with C.L.Terng [26] who conjectured the existence of infinite dimensional symmetric spaces. Important progresses towards their constructions and geometries are achieved by B. Popescu [23], W. Freyn [10], Heintze [13, 14], et al. Now it has been shown that affine Kac-Moody symmetric spaces are tame Fréchet manifolds. In particular let G be the simply connected Lie group with Lie algebra \mathfrak{g} and denote σ be the automorphism of G corresponding to $\sigma \in \text{Aut}(\mathfrak{g})$. Then the loop group

$$L(G, \sigma) = \{g : \mathbb{R} \rightarrow G \mid g \in C^\infty, g(t + 2\pi) = \sigma(g(t)) \quad \forall t\} \quad (1)$$

with point wise multiplication is a Fréchet Lie group with Lie algebra $L(\mathfrak{g}, \sigma)$. The affine Kac-Moody group $\hat{L}(G, \sigma)$ will be a $T^2 = S^1 \times S^1$ bundle over $L(G, \sigma)$. By construction $\hat{L}(G, \sigma)$ is a Fréchet group and it is a Lorentz manifold. It is also well known that $\hat{L}(G, \sigma)/\hat{L}(G, \sigma)^{\hat{\rho}}$ for any involution $\hat{\rho}$ of $\hat{L}(G, \sigma)$ and $L(G, \sigma)/L(G, \sigma)^\rho$ for any involution ρ are tame Fréchet.

The classification of affine Kac-Moody symmetric spaces is essentially equivalent to the classification of involutions of affine Kac-Moody algebras upto conjugation. This has been achieved by a long series of papers by various authors Batra [1], Levstein [19], Kobayashi [17], Rousseau and Messaoud [24]. There is a one-to-one correspondence between real forms of Kac-Moody algebras, involutions. The diagrammatic representation of real forms are the Vogan diagrams (to each real form there is a unique Vogan diagram) which are Dynkin diagrams together with some extra pieces of informations. We have collected/constructed the Vogan diagrams related with untwisted classical algebras $A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ explicitly. From the Vogan diagrams we have obtained the fixed point set as well as the real forms of the algebras and then we have constructed affine Kac-Moody symmetric spaces.

In chapter-2 we have given a brief introduction to affine untwisted Kac-Moody algebras with their involutions, real forms and Vogan diagrams.

In chapter-3 we have explicitly calculated the affine Kac-Moody symmetric spaces with two elementary untwisted Kac-Moody algebras $A_1^{(1)}$ and $A_2^{(1)}$ relating them with their Vogan diagrams to corroborate our technique to construct

*Electronic address: kcpati@nitrkl.ac.in

affine Kac-Moody symmetric spaces. Towards the end of the chapter we have given exhaustive list of all the affine Kac-Moody symmetric spaces together with their real forms and fixed algebras. Chapter-4 contains appendix.

II. PRELIMINARIES

A. Kac-Moody Lie algebras

Let $I = [1, n + 1]$, $n \in \mathbb{N}$, be an interval in \mathbb{N} . A matrix $A = (a_{ij})_{i,j \in I}$ with integer coefficients is called a generalized Cartan matrix if it satisfies the following conditions:

1. $a_{ii} = 2$ for $i = 1, \dots, n + 1$.
2. $a_{ij} \leq 0$ for $i \neq j$.
3. $a_{ij} = 0$ iff $a_{ji} = 0$.

A realization of A is a triple $(\mathfrak{h}, \pi, \tilde{\pi})$, where \mathfrak{h} is a finite-dimensional complex vector space and $\pi = \{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$ and $\tilde{\pi} = \{\check{\alpha}_i\}_{i \in I} \subset \mathfrak{h}$ are indexed subsets in \mathfrak{h}^* and \mathfrak{h} respectively, and they satisfy

1. both $\pi, \tilde{\pi}$ are linearly independent.
2. $\langle \alpha_i, \check{\alpha}_j \rangle = a_{ji}$, for $i, j = 1, \dots, n$.
3. $\text{rank}(A) = 2n - \dim \mathfrak{h}$.

For any $n \times n$ matrix A there exists a unique (upto isomorphism) realization.

Given two matrices A and A' and their realizations $(\mathfrak{h}, \pi, \tilde{\pi})$ and $(\mathfrak{h}', \pi', \tilde{\pi}')$, we obtain a realization of the direct sum of two matrices $(\mathfrak{h} \oplus \mathfrak{h}', \pi \cup \{0\} \cup \{0\} \cup \pi', \tilde{\pi} \cup \{0\} \cup \{0\} \cup \tilde{\pi}')$ which is called direct sum of the realizations.

A matrix A is called decomposable if after reordering of indices A decomposes into a non-trivial direct sum. Otherwise A is called indecomposable.

Let $A = (a_{ij})$ be a generalized Cartan matrix and let $(\mathfrak{h}, \pi, \tilde{\pi})$ be a realization of A . Let $g(A)$ be a complex Lie algebra with generators e_i, f_i for $i = 1 \dots n$ and \mathfrak{h} and the following defining relations:

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} \check{\alpha}_i \\ [h, h'] &= 0 \\ [h, e_i] &= \langle \alpha_i, h \rangle e_i \\ [h, f_i] &= -\langle \alpha_i, h \rangle f_i \end{aligned}$$

and the Serre relations

$$(ad e_i)^{1-a_{ij}} e_j = 0, \quad (ad f_i)^{1-a_{ij}} f_j = 0; \quad \forall i \neq j, \quad (2)$$

The Lie algebra $g = g(A)$ is called a Kac-Moody algebra. The subalgebra \mathfrak{h} of g is called the Cartan subalgebra. The matrix A is the Cartan matrix of g which is of rank n . The elements e_i, f_i for $i = 1 \dots n$, are called Chevalley generators and they generate the subalgebra $g' = [g, g]$ and $g = g' + \mathfrak{h}$.

B. Affine Kac-Moody Algebras

Consider the generalized Cartan matrix A . It is called a Cartan matrix of affine type if

- A is an indecomposable matrix, i.e. after the indices are reordered A cannot be written in the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$.
- There exists a vector $(a_i)_{i=1}^{n+1}$, with a_i all positive such that $A(a_i)_{i=1}^{n+1} = 0$.

Then, the algebra g associated with A is called an affine Kac-Moody algebra. Affine Kac-Moody algebras are of two types untwisted and twisted. In this paper we have confined ourselves to untwisted case only.

C. A realization of non-twisted affine Kac-Moody Lie algebra

Let $L = \mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in t . The residue of the Laurent polynomial $P = \sum_{j \in \mathbb{Z}} c_j t^j$ (where all but finite number of c_j are zero) is $\text{Res}P = c_{-1}$.

Let \mathfrak{g} be a finite-dimensional simple Lie algebra over \mathbb{C} of type X_n , then $L(\mathfrak{g}) = L \otimes \mathfrak{g}$ is an infinite-dimensional Lie algebra with the bracket

$$[P \otimes X, Q \otimes Y] = PQ \otimes [X, Y] \quad P, Q \in L; X, Y \in \mathfrak{g}. \quad (3)$$

Fix a non-degenerate, invariant, symmetric bilinear form (\cdot, \cdot) in \mathfrak{g} and extend this form to an L valued form $(\cdot, \cdot)_t$ on $L(\mathfrak{g})$ by

$$(P \otimes x, Q \otimes y)_t = PQ(x, y) \quad P, Q \in L; x, y \in \mathfrak{g}. \quad (4)$$

The derivation $t^j(d/dt)$ of L extends to $L(\mathfrak{g})$ by

$$t^j \frac{d}{dt}(P \otimes X) = t^j \frac{dP}{dt} \otimes X, \quad P \in L; X \in \mathfrak{g}. \quad (5)$$

Therefore $\psi(a, b) = \text{Res}(\frac{da}{dt}, b)_t$ for $a, b \in L(\mathfrak{g})$ defines a two-cocycle on $L(\mathfrak{g})$.

Now we denote by $\tilde{L}(\mathfrak{g})$ the central extension of the Lie algebra $L(\mathfrak{g})$ associated to the cocycle ψ . Explicitly $\tilde{L}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \mathbb{C}c$ with the bracket

$$[a + \lambda c, b + \mu c] = [a, b] + \psi(a, b)c \quad a, b \in L(\mathfrak{g}); \lambda, \mu \in \mathbb{C}. \quad (6)$$

Finally, denote by $\hat{L}(\mathfrak{g})$ is the Lie algebra which is obtained by adjoining to $\tilde{L}(\mathfrak{g})$ a derivation d which acts on $L(\mathfrak{g})$ as $t \frac{d}{dt}$ and kills c . Explicitly we have $\hat{L}(\mathfrak{g}) = \tilde{L}(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$ with the bracket defined by

$$[t^k \otimes x + \lambda c + \mu d, t^j \otimes y + \lambda_1 c + \mu_1 d] = t^{j+k} \otimes [x, y] + \mu_1 j t^j \otimes y - \mu_1 k t^k \otimes x + k \delta_{j, -k}(x, y)c, \quad (7)$$

where $x, y \in \mathfrak{g}; \lambda, \mu, \lambda_1, \mu_1 \in \mathbb{C}; j, k \in \mathbb{Z}$. This $\hat{L}(\mathfrak{g})$ is a non-twisted affine Kac-Moody Lie algebra associated to the affine matrix A of type $X_n^{(1)}$.

D. Automorphisms and Real forms of non-twisted affine Kac-Moody algebras

Define a group G acting on the algebra g through adjoint representation $Ad : G \rightarrow \text{Aut}(g)$. It is generated by the subgroup U_α for $\alpha \in \pm\pi$ and $Ad(U_\alpha) = \exp(ad(g_\alpha))$.

A maximal ad_g -diagonalizable subalgebra of g is called a Cartan subalgebra. Every Cartan subalgebra of g is $Ad(G)$ -conjugate to the standard Cartan subalgebra \mathfrak{h} . A Borel subalgebra of g is maximal completely solvable subalgebra. It is conjugated by $Ad(G)$ to b^+ and b^- where $b^+ = \mathfrak{h} \oplus \bigoplus_{\alpha > 0} g_\alpha$ and $b^- = \mathfrak{h} \oplus \bigoplus_{\alpha < 0} g_\alpha$. However b^+ and b^- are not conjugated under $Ad(G)$. So there are two conjugacy classes of Borel subalgebra the positive and negative subalgebras.

A real form of g is a algebra $g_{\mathbb{R}}$ over \mathbb{R} such that there exists an isomorphism from g to $g_{\mathbb{R}} \otimes \mathbb{C}$. If we replace \mathbb{C} by \mathbb{R} in definition of g then we obtain a real form of $g_{\mathbb{R}}$ which is called split real form.

An automorphism σ of g is called an involution if $\sigma^2 = Id$. The involution is called semi-linear if $\sigma(\lambda x) = \bar{\lambda} \sigma(x)$ for $\lambda \in \mathbb{C}$ and $x \in g$. A real form of g correspondences to a semi-linear involution of g . A linear or semi-linear automorphism σ of g is said to be of first kind if $\sigma(b^+)$ is $Ad(G)$ -conjugate to b^+ and it is of second kind if $\sigma(b^+)$ is $Ad(G)$ -conjugate to b_- . Any automorphism of g is either an automorphism of first kind (type 1) or an automorphism of second kind (type 2).

Let $g_{\mathbb{R}}$ be a real form of g . Fix an isomorphism from g to $g_{\mathbb{R}} \otimes \mathbb{C}$. Then the Galois group $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R})$ acts on g and the corresponding group G . Then $g_{\mathbb{R}}$ can be identified with the fixed point set g^{Γ} . If Γ consists of first kind automorphism then we say $g_{\mathbb{R}}$ is almost split, otherwise if the non-trivial element of Γ is of second kind automorphism then we say $g_{\mathbb{R}}$ is almost compact(non-compact). Denote the group of \mathbb{C} -linear or semilinear automorphisms of g as $\text{Aut}_{\mathbb{R}}(g)$. The group $\text{Aut}(g)$ is normal in $\text{Aut}_{\mathbb{R}}(g)$ and of index 2. A semilinear automorphism of order 2 of g is called a semiinvolution of g .

Definition II.1. Let σ' be a semi-involution of g of second kind and let $g_{\mathbb{R}} = g^{\sigma'}$ be the corresponding almost compact real form. A Cartan semi-involution ω' which commutes with σ' is called a Cartan semi-involution for σ' or $g_{\mathbb{R}}$. The involution $\sigma = \sigma' \omega'$ is called a Cartan involution of σ' and also the restriction $\omega_{\mathbb{R}'}$ of σ to $g_{\mathbb{R}}$ is called Cartan involution for $g_{\mathbb{R}}$.

The algebra of fixed points $\mathfrak{t}_\sigma = g_\mathbb{R}^\sigma$ is called a maximal compact subalgebra of $g_\mathbb{R}$. Now we have the Cartan decomposition $g_\mathbb{R} = \mathfrak{t}_\sigma \oplus \mathfrak{p}_\sigma$ and $\mathfrak{t}_1 = \mathfrak{t}_\sigma \oplus i\mathfrak{p}_\sigma$ where \mathfrak{p}_σ is the eigenspace of $\omega'_\mathbb{R}$ for eigen value -1 . Let t_0 be a maximal abelian subspace of \mathfrak{t}_σ . Then $\mathfrak{h}_\sigma = Z_{g_\mathbb{R}}(t_0)$ is a σ -stable Cartan subalgebra of the almost compact real form $g_\mathbb{R}$ of the form $\mathfrak{h}_\sigma = t_0 \oplus a_0$ with $a_0 \subseteq \mathfrak{p}_\sigma$.

Definition II.2. A σ -stable Cartan subalgebra $\mathfrak{h}_\sigma = t_0 \oplus a_0$ with $t_0 \subseteq \mathfrak{t}_\sigma$ and $a_0 \subseteq \mathfrak{p}_\sigma$ of an almost compact real form $g_\mathbb{R}$ is maximally compact if the dimension of t_0 is as large as possible and it is maximally non-compact if the dimension of a_0 is as large as possible.

A maximally compact Cartan subalgebra \mathfrak{h}_σ of an almost compact real form $g_\mathbb{R}$ has the property that all the roots are real on a_0 and imaginary on t_0 . One says that a root is real if it takes real value on $\mathfrak{h}_\sigma = t_0 \oplus a_0$, i.e. vanishes on t_0 . It is imaginary if it takes imaginary value on \mathfrak{h}_σ , i.e. vanishes on a_0 and complex otherwise.

E. Classification of Real forms

Under $Aut(g)$ there is a one-one correspondence between the conjugacy classes of involutions(linear) of second kind of g and the conjugacy classes of almost split real forms of g . Again there is a bijection between the conjugacy classes under $Aut(g)$ of semi-involution of second kind and conjugacy classes of involution of first kind. Thus one obtain under $Aut(g)$ a one-to-one correspondence between conjugacy classes of linear involutions of first kind(including identity) and the the conjugacy classes of almost compact real forms of g . The compact real form is unique and corresponds to the identity.

Let \mathfrak{h}_σ be a σ -stable Cartan subalgebra of $g_\mathbb{R}$. Then there are no real roots iff \mathfrak{h}_σ is maximally compact.

Let $g_\mathbb{R}$ be almost compact real form of g corresponding to the semi-involution of the second kind σ' of g . Let σ be the Cartan involution of $g_\mathbb{R}$ and let $g_\mathbb{R} = \mathfrak{t}_\sigma \oplus \mathfrak{p}_\sigma$ be the corresponding Cartan decomposition[3, 12, 28]. Let t_0 be maximal abelian subspace of \mathfrak{t}_σ . Then $\mathfrak{h}_\sigma = Z_{g_\mathbb{R}}(t_0)$ is a σ -stable Cartan subalgebra of $g_\mathbb{R}$ of the form $\mathfrak{h}_\sigma = t_0 \oplus a_0$ with $a_0 \subseteq \mathfrak{p}_\sigma$. This \mathfrak{h}_σ is a maximally Cartan subalgebra of $g_\mathbb{R}$ because t_0 is as large as possible.

For any root α , $\sigma(\alpha)$ is the root $\sigma\alpha(H) = \alpha(\sigma^{-1}H)$. If α is imaginary then $\sigma(\alpha) = \alpha$ and α vanishes on a_0 . Thus g_α is σ -stable and we have $g_\alpha = (g_\alpha \cap \mathfrak{t}) \oplus (g_\alpha \cap \mathfrak{p})$. Again $\dim(g_\alpha) = 1$, so $g_\alpha \subseteq \mathfrak{t}$ or $g_\alpha \subseteq \mathfrak{p}$. An imaginary root α is called compact if $g_\alpha \subseteq \mathfrak{t}$ and is non-compact if $g_\alpha \subseteq \mathfrak{p}$.

Theorem II.3. (Theorem 45 [10]) Let g be an complex affine Kac-Moody algebra and \mathcal{C} be a real form of it which is compact type. The conjugacy classes of real forms of non compact type of g are in bijection with the conjugacy classes of involutions on \mathcal{C} . The correspondence is given by $\mathcal{C} = K \oplus P \mapsto K \oplus iP$ where K and P are the \pm -eigen spaces for the involution.

However every real form is either of compact type or of non-compact type, a mixed type is not possible.

Lemma II.4. (Lemma 47 [10]) Let $g_\mathbb{R}$ be a real form of non-compact type. Let $g_\mathbb{R} = K \oplus P$ be Cartan decomposition. The Cartan Killing form is negative definite on K and positive definite on P .

F. Vogan diagrams

For classification of real forms of affine Kac-Moody algebra there are two main approaches: One focuses the maximal non-compact Cartan subalgebra that leads to Satake diagrams [27]. The other one is on maximal compact Cartan subalgebra that leads to Vogan diagrams [2, 6, 18].

Let almost compact real form $g_\mathbb{R}$ of g and σ be the Cartan involution on $g_\mathbb{R}$ leading to the Cartan decomposition $g_\mathbb{R} = \mathfrak{t}_\sigma \oplus \mathfrak{p}_\sigma$. Let \mathfrak{h}_σ be the maximally compact σ -stable Cartan subalgebra of $g_\mathbb{R}$ with complexification $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. Let us denote $\Delta = \Delta(g, \mathfrak{h})$ be the set of roots of g with respect to \mathfrak{h} . This set doesn't contain any real root as \mathfrak{h}_σ is assumed to be maximally compact. From Δ we choose a positive system Δ^+ that takes it_0 before a . since σ is $+1$ on \mathfrak{t}_σ and -1 on a_0 and since there are no real roots $\sigma(\Delta^+) = \Delta^+$. Therefore σ permutes the simple roots. It fixes the simple roots that are imaginary and permutes in 2-cycles the simple roots that are complex.

Definition II.5. By Vogan diagram of the triple $(g_\mathbb{R}, \mathfrak{h}_\sigma, \Delta^+)$ we mean the Dynkin diagram of Δ^+ with the 2-element orbits under σ labelled an arrow and with the 1-element orbit painted or not depending upon whether the corresponding imaginary simple root is non-compact or compact.

Every Vogan diagram represents an almost compact(non-compact) real form of some affine Kac-Moody Lie algebra. Two diagrams may represent isomorphic algebras and in that case the diagrams are equivalent. So the classification of Vogan diagram gives rise to the classification of almost compact real form of affine Kac-Moody Lie algebra.

The equivalence of Vogan diagram is defined as the equivalence relation generated by the following two operations:

1. Applications of an automorphism of the Dynkin diagram.
2. Change in the positive system by reflection in a simple, non-compact root, i.e. by a vertex which is colored in the Vogan diagram.

As a consequence of reflection by a simple non-compact root α , the rules for single and triple lines is that we have α colored and its immediate neighbour is changed to the opposite color. The rule for double line is that if α is the smaller root, then there is no change in the color of immediate neighbour, but we leave α colored. If α is a bigger root, then we leave α colored and the immediate neighbour is changed to the opposite color.

If two Vogan diagrams aren't equivalent to each other, then they are called non-equivalent.

Definition II.6. An abstract Vogan diagram is an irreducible abstract Dynkin diagram of non-twisted affine Kac-Moody Lie algebra with two additional piece of structure as follows:

1. One is an automorphism of order 1 or 2 of the diagram, which is indicated by labelling the 2-element orbits.
2. Second one is a subset of 1-element orbits which is to be indicated by pointing the vertices corresponding to the members of the subset.

Every Vogan diagram is an abstract Vogan diagram. It is always convenient to represent equivalence class of Vogan diagrams with minimum number of vertices painted. We have Borel Seibenthal theorem for affine Kac-Moody algebras [7] which states that

Theorem II.7. *Every equivalence class of Vogan diagram has a representative with atmost two vertices painted.*

Some more important results:

Theorem II.8. *If an abstract Vogan diagram for an non-twisted affine Kac-Moody Lie algebra is given, then there exists an almost compact real form of a non-twisted affine Kac-Moody Lie algebra such that the given diagram is the Vogan diagram of this almost compact real form.*

Theorem II.9. *If two almost compact real forms of a non-twisted affine Kac-Moody Lie algebra g have equivalent Vogan diagram then they are isomorphic.*

III. AFFINE KAC-MOODY SYMMETRIC SPACE

In this chapter we briefly review with the definition and geometry of the affine Kac-Moody symmetric spaces [10, 13, 23] with explicit determination of affine Kac-Moody symmetric spaces associated with $A_1^{(1)}$ and $A_2^{(1)}$.

Definition III.1. A tame Fréchet manifold M with a weak metric having a Levi-civita connection is called a symmetric space, iff $\forall p \in M$ there is an involution isometry ρ_p , such that p is an isolated fixed point of ρ_p .

Definition III.2. An(affine) Kac-Moody symmetric space M is a tame Fréchet Lorentz symmetric space such that its isometry group $I(M)$ contains a transitive subgroup isomorphic to an affine geometric Kac-Moody group \mathcal{H} and the intersection of the isotropy group of a point with \mathcal{H} is a loop group of compact type.

Theorem III.3. *(Affine Kac-Moody symmetric spaces of compact type)*

Both the Kac-Moody group $\widehat{MG}_{\mathbb{R}}^{\sigma}$ equipped with its Ad-invariant metric, and the quotient space $X = \widehat{MG}_{\mathbb{R}}^{\sigma}/Fix(\rho_)$ equipped with its $Ad(Fix(\rho_*))$ -invariant metric are tame Fréchet symmetric spaces of the compact type with respect to their Ad-invariant metric. Their curvatures satisfy $\langle R(X, Y)X, Y \rangle \geq 0$.*

Theorem III.4. *(Affine Kac-Moody symmetric spaces of non-compact type)*

Both quotient spaces $X = \widehat{MG}_{\mathbb{C}}^{\sigma}/\widehat{MG}_{\mathbb{R}}^{\sigma}$ and $X = H/Fix(\rho_)$ where H is a non-compact real form of $\widehat{MG}_{\mathbb{C}}^{\sigma}$ with their Ad-invariant metric are tame Fréchet symmetric spaces of non-compact type. Their curvatures satisfy $\langle R(X, Y)X, Y \rangle \leq 0$. Furthermore Kac-Moody symmetric spaces of the non compact type are diffeomorphic to vector space.*

Theorem III.5. *(Duality)*

Affine Kac-Moody symmetric spaces of compact type are dual to the Affine Kac-Moody symmetric spaces of non-compact type and vice versa.

We can summarize all the results we have discussed so far as follows:

There is an one-one correspondence between the conjugacy classes of involution of second kind of affine Kac-Moody algebra g and almost split real form and also between the conjugacy classes of involution of first kind and almost compact real form of g . Again to each real form there is a unique Vogan diagram. On the other hand all real forms are of two types: compact, non-compact and the conjugacy classes of real forms of non-compact type of g are in bijection with the conjugacy classes of involution on the compact real form (\mathcal{C}). So the affine Kac-Moody symmetric spaces can be classified using compact real form and involution of second kind. Now we can conclude that classification affine Kac-Moody symmetric spaces are intimately linked with classification of Vogan diagram. Starting with the compact real form \mathcal{C} we can construct all the various non-compact real forms by applying involutive automorphisms to \mathcal{C} followed by Weyl unitary trick.

A. Affine Kac-Moody symmetric spaces associated with $A_1^{(1)}$

The chevelley generators for $A_1^{(1)}$ are given by:

$$\left\{ e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e, f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = f, e_2 = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} = tf, f_2 = \begin{pmatrix} 0 & t^{-1} \\ 0 & 0 \end{pmatrix} = t^{-1}e, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

The Cartan involution of $A_1^{(1)}$ is the following

$$\begin{array}{ll} t^n e \mapsto -t^{-n} f & it^n e \mapsto it^{-n} f \\ t^n f \mapsto -t^{-n} e & it^n f \mapsto it^{-n} e \\ t^n h \mapsto -t^{-n} h & it^n h \mapsto it^{-n} h \\ c \mapsto -c & ic \mapsto ic \\ d \mapsto -d & id \mapsto id \end{array}$$

As a result the compact form is generated by: $\{(t^n e - t^{-n} f), i(t^n e + t^{-n} f), (t^n f - t^{-n} e), i(t^n f + t^{-n} e), (t^n h - t^{-n} h), i(t^n h + t^{-n} h) | n \in \mathbb{Z}\} \oplus \mathbb{R}ic \oplus \mathbb{R}id$. Explicitly the compact real form is given by $\mathcal{C}(t) = \sum_{n \in \mathbb{Z}} [a_1^{(n)}(t^n e - t^{-n} f) + ia_2^{(n)}(t^n e + t^{-n} f) + a_3^{(n)}(t^n f - t^{-n} e) + ia_4^{(n)}(t^n f + t^{-n} e) + a_5^{(n)}(t^n h - t^{-n} h) + ia_6^{(n)}(t^n h + t^{-n} h)] \oplus \mathbb{R}ic \oplus \mathbb{R}id$ which is equal to the following matrix

$$\mathcal{C}(t) = \sum_{n \in \mathbb{Z}} \begin{pmatrix} a_5^{(n)}(t^n - t^{-n}) + ia_6^{(n)}(t^n + t^{-n}) & t^n(a_1^{(n)} + ia_2^{(n)}) + t^{-n}(-a_3^{(n)} + ia_4^{(n)}) \\ t^{-n}(-a_1^{(n)} + ia_2^{(n)}) + t^n(a_3^{(n)} + ia_4^{(n)}) & a_5^{(n)}(t^{-n} - t^n) - ia_6^{(n)}(t^n + t^{-n}) \end{pmatrix} \oplus \mathbb{R}ic \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{R}id \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

this is a skew hermitian matrix with trace zero, which is identified as $su^{(1)}(2)$. The Vogan diagram associated with this real form is given by the following:

$$\alpha_0 \circ \longleftrightarrow_{\alpha_1} \circ$$

Now the general form [8] of an involutive automorphism associated with a affine Kac-Moody algebra is given as: for type 1(a) automorphism

$$\sigma(\mathcal{C}(t)) = U(t)\mathcal{C}(ut)U(t)^{-1} + \frac{1}{\gamma} \text{Res} \left\{ \text{tr} \left(U(t)^{-1} \frac{dU(t)}{dt} \mathcal{C}(u(t)) \right) \right\} c. \quad (8)$$

and for type 1(b) automorphism

$$\sigma(\mathcal{C}(t)) = U(t)(-\tilde{\mathcal{C}}(ut))U(t)^{-1} + \frac{1}{\gamma} \text{Res} \left\{ \text{tr} \left(U(t)^{-1} \frac{dU(t)}{dt} (-\tilde{\mathcal{C}}(ut)) \right) \right\} c. \quad (9)$$

But the conjugacy class of type 1(b) automorphisms with $u = 1$ and $u = -1$ correspond to some automorphisms of type 1(a) with $u = 1$ and $u = -1$ respectively. Also we shall like to mention that type 2(a) and type 2(b) automorphisms are obtained by composing type 1(a) and 1(b) with Cartan involution respectively. Action of σ on c is $\sigma(c) = \mu c$, however for 1(a) automorphism $\mu = 1$. Now $\sigma(d) = \mu \Phi(U(t)) + \xi c + \mu d$ where $\Phi(U(t))$ is the $d_\Gamma \times d_\Gamma$

matrix that depends upon $U(t)$ as below,

$$\Phi(U(t)) = \left\{ -t \frac{dU(t)}{dt} U(t)^{-1} + \frac{1}{d_\Gamma} \text{tr} \left(t \frac{dU(t)}{dt} U(t)^{-1} \right) I \right\}$$

and for our cases $\sigma(d) = d$ except the case-III of $A_1^{(1)}$ where

$$\sigma(d) = \begin{pmatrix} t^2/2 & 0 \\ 0 & -t^2/2 \end{pmatrix} + d.$$

Case-I: If

$$U(t) = U(t)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, u = 1, \quad \xi = 0, \quad (10)$$

then under the automorphism (8) with (10) a general matrix with block matrices A, B, C, D transforms as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}.$$

Hence the fixed subalgebra K of $\mathcal{C}(t)$ is given by

$$K = \sum_{n \in \mathbb{Z}} \begin{pmatrix} a_5^{(n)}(t^n - t^{-n}) + ia_6^{(n)}(t^n + t^{-n}) & 0 \\ 0 & a_5^{(n)}(t^{-n} - t^n) - ia_6^{(n)}(t^n + t^{-n}) \end{pmatrix}$$

and

$$P = \sum_{n \in \mathbb{Z}} \begin{pmatrix} 0 & t^n(a_1^{(n)} + ia_2^{(n)}) + t^{-n}(-a_3^{(n)} + ia_4^{(n)}) \\ t^{-n}(-a_1^{(n)} + ia_2^{(n)}) + t^n(a_3^{(n)} + ia_4^{(n)}) & 0 \end{pmatrix}.$$

Thus $K + iP = \sum_{n \in \mathbb{Z}} \begin{pmatrix} a_5^{(n)}(t^n - t^{-n}) + ia_6^{(n)}(t^n + t^{-n}) & t^n(-a_2^{(n)} + ia_1^{(n)}) + t^{-n}(-ia_3^{(n)} - a_4^{(n)}) \\ t^{-n}(-ia_1^{(n)} - a_2^{(n)}) + t^n(ia_3^{(n)} - a_4^{(n)}) & a_5^{(n)}(t^{-n} - t^n) - ia_6^{(n)}(t^n + t^{-n}) \end{pmatrix}$ Hence the non-compact real form is $K + iP \oplus \mathbb{R}ic \oplus \mathbb{R}id \in su_1^{(1)}(1, 1)$ and the corresponding Vogan diagram is given by

$$\alpha_0 \longleftrightarrow \alpha_1$$

and the two affine Kac-Moody symmetric spaces are

$$SU_1^{(1)}(1, 1)/S_1^{(1)}(U_1 \times U_1), \quad SU^{(1)}(1 + 1)/S_1^{(1)}(U_1 \times U_1). \quad (11)$$

Case II: Similarly if

$$U(t) = U(t)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = -1, \quad \xi = 0 \quad (12)$$

then under the automorphism(8) with (12) the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ transforms as

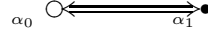
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} -A & B \\ C & -D \end{pmatrix}.$$

We observe when n is an even integer it reduces to case I, giving same real form. But n is odd then we have

$$K = \sum_{n \in \mathbb{Z}} \begin{pmatrix} 0 & t^n(a_1^{(n)} + ia_2^{(n)}) + t^{-n}(-a_3^{(n)} + ia_4^{(n)}) \\ t^{-n}(-a_1^{(n)} + ia_2^{(n)}) + t^n(a_3^{(n)} + ia_4^{(n)}) & 0 \end{pmatrix}$$

and

$$P = \sum_{n \in \mathbb{Z}} \begin{pmatrix} a_5^{(n)}(t^n - t^{-n}) + ia_6^{(n)}(t^n + t^{-n}) & 0 \\ 0 & t^{-n}(-a_1^{(n)} + ia_2^{(n)}) + t^n(a_3^{(n)} + ia_4^{(n)}) \end{pmatrix}.$$



$$K + iP = \sum_{n \in \mathbb{Z}} \begin{pmatrix} ia_5^{(n)}(t^n - t^{-n}) - a_6^{(n)}(t^n + t^{-n}) & t^n(a_1^{(n)} + ia_2^{(n)}) + t^{-n}(-a_3^{(n)} + ia_4^{(n)}) \\ t^{-n}(-a_1^{(n)} + ia_2^{(n)}) + t^n(a_3^{(n)} + ia_4^{(n)}) & ia_5^{(n)}(t^{-n} - t^n) + a_6^{(n)}(t^n + t^{-n}) \end{pmatrix} \quad \text{Hence the non-}$$

compact real form is $K + iP \oplus \mathbb{R}ic \oplus \mathbb{R}id \in su_{-1}^{(1)}(1, 1)$. The corresponding Vogan diagram is the given by
The affine Kac-Moody symmetric spaces are:

$$SU_{-1}^{(1)}(1, 1)/S_{-1}^{(1)}(U_1 \times U_1), \quad SU^{(1)}(1 + 1)/S_{-1}^{(1)}(U_1 \times U_1). \quad (13)$$

Case III: Now consider

$$U(t) = \begin{pmatrix} 0 & 1 \\ -t & 0 \end{pmatrix}, \quad u = 1, \quad \xi = -1, \quad U(t)^{-1} = \begin{pmatrix} 0 & -t \\ 1 & 0 \end{pmatrix}. \quad (14)$$

So now under the automorphism (8) with (14) a general matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ transform as,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} D & -Ct^{-1} \\ -Bt & A \end{pmatrix}. \quad (15)$$

Then from a simple mathematical manipulation we observe that in this case

$$K = \begin{pmatrix} A + D & B - Ct^{-1} \\ C - Bt & D + A \end{pmatrix}, \quad P = \begin{pmatrix} A - D & B + Ct^{-1} \\ C + Bt & D - A \end{pmatrix} \quad (16)$$

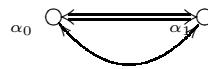
Now putting the values of A, B, C and D in K we get

$$K = \frac{1}{2} \sum_{n \in \mathbb{Z}} \begin{pmatrix} 0 & t^n(a_1^{(n)} + ia_2^{(n)}) + t^{-n}(-a_3^{(n)} + ia_4^{(n)}) + t^{-(n+1)}(a_1^{(n)} - ia_2^{(n)}) - (a_3^{(n)} + ia_4^{(n)})t^{n-1} \\ t^{-n}(-a_1^{(n)} + ia_2^{(n)}) + t^n(a_3^{(n)} + ia_4^{(n)}) & 0 \\ +t^{n+1}(-a_1^{(n)} - ia_2^{(n)}) + (a_3^{(n)} - ia_4^{(n)})t^{-n+1} & 0 \end{pmatrix}$$

with K satisfies $K^* + K = 0$ and hence K is identified as $so^{(1)}(2)$. So we have $K + iP =$

$$\frac{1}{2} \sum_{n \in \mathbb{Z}} \begin{pmatrix} a_5^{(n)}(t^n - t^{-n}) + ia_6^{(n)}(t^n + t^{-n}) & t^n(a_1^{(n)} + ia_2^{(n)}) + t^{-n}(-a_3^{(n)} + ia_4^{(n)}) + t^{-(n+1)}(a_1^{(n)} - ia_2^{(n)}) - (a_3^{(n)} + ia_4^{(n)})t^{n-1} \\ t^{-n}(-a_1^{(n)} + ia_2^{(n)}) + t^n(a_3^{(n)} + ia_4^{(n)}) + t^{n+1}(-a_1^{(n)} - ia_2^{(n)}) + (a_3^{(n)} - ia_4^{(n)})t^{-n+1} & a_5^{(n)}(-t^n + t^{-n}) - ia_6^{(n)}(t^n + t^{-n}) \end{pmatrix} \oplus \mathbb{R}ic \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{R}id \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is identified as $sl^{(1)}(2, \mathbb{R})$ and the Vogan diagram is



Hence the affine Kac-Moody symmetric spaces are

$$SU^{(1)}(2)/SO^{(1)}(2), \quad SL^{(1)}(2, \mathbb{R})/SO^{(1)}(2). \quad (17)$$

B. Affine Kac Moody symmetric space associated with $A_2^{(1)}$

The Chevelley generators of $A_2^{(1)}$ are: $\left\{ e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t & 0 & 0 \end{pmatrix}, f_3 = \begin{pmatrix} 0 & 0 & t^{-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$. The Cartan involution of $A_2^{(1)}$ is the following:

$$\begin{array}{ll}
t^n e_1 \mapsto -t^{-n} f_1 & it^n e_1 \mapsto it^{-n} f_1 \\
t^n e_2 \mapsto -t^{-n} f_2 & it^n e_2 \mapsto it^{-n} f_2 \\
t^n e_3 \mapsto -t^{-n} f_3 & it^n e_3 \mapsto it^{-n} f_3 \\
t^n f_1 \mapsto -t^{-n} e_1 & it^n f_1 \mapsto it^{-n} e_1 \\
t^n f_2 \mapsto -t^{-n} e_2 & it^n f_2 \mapsto it^{-n} e_2 \\
t^n f_3 \mapsto -t^{-n} e_3 & it^n f_3 \mapsto it^{-n} e_3 \\
t^n h_1 \mapsto -t^{-n} h_1 & it^n h_1 \mapsto it^{-n} h_1 \\
t^n h_2 \mapsto -t^{-n} h_2 & it^n h_2 \mapsto it^{-n} h_2 \\
t^n h_3 \mapsto -t^{-n} h_3 & it^n h_3 \mapsto it^{-n} h_3 \\
c \mapsto -c & ic \mapsto ic \\
d \mapsto -d & id \mapsto id
\end{array}$$

Hence the Compact form is generated by $\{e_1 t^n - f_1 t^{-n}, i(e_1 t^n + f_1 t^{-n}), h_1 t^n - h_1 t^{-n}, i(h_1 t^n + h_1 t^{-n}), e_2 t^n - f_2 t^{-n}, i(e_2 t^n + f_2 t^{-n}), h_2 t^n - h_2 t^{-n}, i(h_2 t^n + h_2 t^{-n}), e_3 t^n - f_3 t^{-n}, i(e_3 t^n + f_3 t^{-n}), h_3 t^n - h_3 t^{-n}, i(h_3 t^n + h_3 t^{-n}) \mid n \in \mathbb{Z}\} \oplus \text{Ric} \oplus \text{Rid}$. So, the compact form $\mathcal{C}(t)$ is

$$\sum_{n \in \mathbb{Z}} \begin{pmatrix} A_{11}^{(n)} & A_{12}^{(n)} & A_{13}^{(n)} \\ A_{21}^{(n)} & A_{22}^{(n)} & A_{23}^{(n)} \\ A_{31}^{(n)} & A_{32}^{(n)} & A_{33}^{(n)} \end{pmatrix} \oplus \text{Ric} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus \text{Rid} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

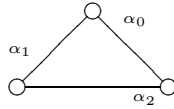
where

$$\begin{aligned}
A_{11}^{(n)} &= a_3^{(n)}(t^n - t^{-n}) + ia_4^{(n)}(t^n + t^{-n}) - a_{11}^{(n)}(t^n - t^{-n}) - ia_{12}^{(n)}(t^n + t^{-n}) \\
A_{12}^{(n)} &= (a_1^{(n)} + ia_2^{(n)})t^n \\
A_{13}^{(n)} &= -a_9^{(n)}t^{-(n+1)} + ia_{10}^{(n)}t^{-(n+1)} \\
A_{21}^{(n)} &= (-a_1^{(n)} + ia_2^{(n)})t^{-n} \\
A_{22}^{(n)} &= a_3^{(n)}(-t^n + t^{-n}) - ia_4^{(n)}(t^n + t^{-n}) + a_7^{(n)}(t^n - t^{-n}) + ia_8^{(n)}(t^n + t^{-n}) \\
A_{23}^{(n)} &= (a_5^{(n)} + ia_6^{(n)})t^n \\
A_{31}^{(n)} &= (ia_{10}^{(n)} + a_9^{(n)})t^{n+1} \\
A_{32}^{(n)} &= (-a_5^{(n)} + ia_6^{(n)})t^{-n} \\
A_{33}^{(n)} &= a_7^{(n)}(-t^n + t^{-n}) - ia_8^{(n)}(t^n + t^{-n}) + a_{11}^{(n)}(t^n - t^{-n}) + ia_{12}^{(n)}(t^n + t^{-n})
\end{aligned}$$

This matrix is in the form:

$$\begin{pmatrix} (A)_{2 \times 2} & (B)_{2 \times 1} \\ (-B^*)_{1 \times 2} & (C)_{1 \times 1} \end{pmatrix}, \quad (18)$$

which is a skew hermitian matrix with trace zero it is identified as $su^{(1)}(3)$. The Vogan diagram is



Now proceeding similarly as $A_1^{(1)}$ case and taking different cases we have,

Case I:

$$U(t) = U(t)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad u = 1, \quad \xi = 0. \quad (19)$$

Under the automorphism (8) with (19) the matrix transforms as

$$\begin{pmatrix} A & B \\ (-B^*) & C \end{pmatrix} \longrightarrow \begin{pmatrix} A & -B \\ B^* & C \end{pmatrix} \quad (20)$$

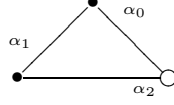
Here

$$K = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}, \quad P = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \quad (21)$$

The decomposition of K as in (??) shows that is isomorphic to $su_1^{(1)}(2) \times c_0 \times su_1^{(1)}(1)$ where c_0 is the center of K . Now $K + iP$ is

$$\begin{pmatrix} (Z_1)_{2 \times 2} & (Z_2)_{2 \times 1} \\ (Z_2^*)_{1 \times 2} & (Z_3)_{1 \times 1} \end{pmatrix} \quad (22)$$

with $Z_2 = iB$ is a 2×2 matrix, $Z_1 = A$ is a 2×2 skew hermitian matrix and $Z_3 = C$ is a 1×1 skew hermitian matrix and also satisfies $Tr Z_1 + Tr Z_3 = 0$. Hence the non-compact real form is $K + iP \oplus \mathbb{R}ic \oplus \mathbb{R}id \in su_1^{(1)}(2, 1)$. The Vogan diagram is



Thus the corresponding symmetric spaces are:

$$SU_1^{(1)}(2, 1)/S_1^{(1)}(U_2 \times U_1), \quad SU^{(1)}(2 + 1)/S_1^{(1)}(U_2 \times U_1). \quad (23)$$

Case II:

$$U(t) = U(t)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad u = -1, \quad \xi = 0. \quad (24)$$

Under the automorphism (8) with (24) the matrix transforms as

$$\begin{pmatrix} A & B \\ -B^* & C \end{pmatrix} \rightarrow \begin{pmatrix} -A & B \\ -B^* & -C \end{pmatrix}. \quad (25)$$

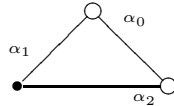
Here

$$K = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}, \quad P = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \quad (26)$$

Thus $K + iP$ is

$$\begin{pmatrix} (Z_1)_{2 \times 2} & (Z_2)_{2 \times 1} \\ (-Z_2^*)_{1 \times 2} & (Z_3)_{1 \times 1} \end{pmatrix} \quad (27)$$

with $Z_2 = iB$ is a 2×1 matrix, $Z_1 = A$ is a 2×2 skew hermitian matrix and $Z_3 = C$ is a 1×1 skew hermitian matrix and also satisfies $Tr Z_1 + Tr Z_3 = 0$. Hence the non-compact real form is $K + iP \oplus \mathbb{R}ic \oplus \mathbb{R}id \in su_{-1}^{(1)}(2, 1)$. The Vogan diagram is



Thus the corresponding symmetric spaces are:

$$SU_{-1}^{(1)}(2, 1)/S_{-1}^{(1)}(U_2 \times U_1), \quad SU^{(1)}(2 + 1)/S_{-1}^{(1)}(U_2 \times U_1). \quad (28)$$

Case III:

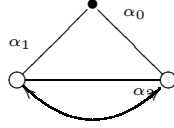
$$U(t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad u = 1, \quad \xi = 0, \quad U(t)^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (29)$$

Under the automorphism (8) with (29) the matrix transforms as

$$\sum_{n \in \mathbb{Z}} \begin{pmatrix} A_{11}^{(n)} & A_{12}^{(n)} & A_{13}^{(n)} \\ A_{21}^{(n)} & A_{22}^{(n)} & A_{23}^{(n)} \\ A_{31}^{(n)} & A_{32}^{(n)} & A_{33}^{(n)} \end{pmatrix} \rightarrow \sum_{n \in \mathbb{Z}} \begin{pmatrix} A_{33}^{(n)} & A_{32}^{(n)} & A_{31}^{(n)} \\ A_{23}^{(n)} & A_{22}^{(n)} & A_{21}^{(n)} \\ A_{13}^{(n)} & A_{12}^{(n)} & A_{11}^{(n)} \end{pmatrix}. \quad (30)$$

$$\begin{aligned}
K &= \frac{1}{2} \sum_{n \in \mathbb{Z}} \begin{pmatrix} A_{11}^{(n)} + A_{33}^{(n)} & A_{12}^{(n)} + A_{32}^{(n)} & A_{13}^{(n)} + A_{31}^{(n)} \\ A_{21}^{(n)} + A_{23}^{(n)} & A_{22}^{(n)} & A_{23}^{(n)} + A_{21}^{(n)} \\ A_{31}^{(n)} + A_{13}^{(n)} & A_{32}^{(n)} + A_{12}^{(n)} & A_{33}^{(n)} + A_{11}^{(n)} \end{pmatrix} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \begin{pmatrix} (\widehat{A}_{11}^{(n)})_+ & (\widehat{A}_{12}^{(n)})_+ & (\widehat{A}_{13}^{(n)})_+ \\ (\widehat{A}_{21}^{(n)})_+ & (\widehat{A}_{22}^{(n)})_+ & (\widehat{A}_{23}^{(n)})_+ \\ (\widehat{A}_{31}^{(n)})_+ & (\widehat{A}_{32}^{(n)})_+ & (\widehat{A}_{33}^{(n)})_+ \end{pmatrix} \\
\text{and } P &= \frac{1}{2} \sum_{n \in \mathbb{Z}} \begin{pmatrix} A_{11}^{(n)} - A_{33}^{(n)} & A_{12}^{(n)} - A_{32}^{(n)} & A_{13}^{(n)} - A_{31}^{(n)} \\ A_{21}^{(n)} - A_{23}^{(n)} & 0 & A_{23}^{(n)} - A_{21}^{(n)} \\ A_{31}^{(n)} - A_{13}^{(n)} & A_{32}^{(n)} - A_{12}^{(n)} & A_{33}^{(n)} - A_{11}^{(n)} \end{pmatrix} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \begin{pmatrix} (\widehat{A}_{11}^{(n)})_- & (\widehat{A}_{12}^{(n)})_- & (\widehat{A}_{13}^{(n)})_- \\ (\widehat{A}_{21}^{(n)})_- & 0 & (\widehat{A}_{23}^{(n)})_- \\ (\widehat{A}_{31}^{(n)})_- & (\widehat{A}_{32}^{(n)})_- & (\widehat{A}_{33}^{(n)})_- \end{pmatrix}. \text{ Thus } K + iP \text{ is} \\
& \frac{1}{2} \sum_{n \in \mathbb{Z}} \begin{pmatrix} (\widehat{A}_{11}^{(n)})_+ + i(\widehat{A}_{11}^{(n)})_- & (\widehat{A}_{12}^{(n)})_+ + i(\widehat{A}_{12}^{(n)})_- & (\widehat{A}_{13}^{(n)})_+ + i(\widehat{A}_{13}^{(n)})_- \\ (\widehat{A}_{21}^{(n)})_+ + i(\widehat{A}_{21}^{(n)})_- & (\widehat{A}_{22}^{(n)})_+ & (\widehat{A}_{23}^{(n)})_+ + i(\widehat{A}_{23}^{(n)})_- \\ (\widehat{A}_{31}^{(n)})_+ + i(\widehat{A}_{31}^{(n)})_- & (\widehat{A}_{32}^{(n)})_+ + i(\widehat{A}_{32}^{(n)})_- & (\widehat{A}_{33}^{(n)})_+ + i(\widehat{A}_{33}^{(n)})_- \end{pmatrix} \quad (31)
\end{aligned}$$

such that trace of this matrix is zero. Hence the non-compact real form is $K + iP \oplus \text{Ric} \oplus \text{Rid} \in \mathfrak{sl}_1^{(1)}(3, \mathbb{R})$. Here $K = \mathfrak{so}_1^{(1)}(3)$. The Vogan diagram is



Thus the corresponding symmetric spaces are

$$SL_1^{(1)}(3, \mathbb{R})/SO_1^{(1)}(3), \quad SU^{(1)}(3)/SO_1^{(1)}(3). \quad (32)$$

Case IV:

$$U(t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad u = -1, \quad \xi = 0, \quad U(t)^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad (33)$$

Under the automorphism (8) with (33) the matrix transforms as

$$\sum_{n \in \mathbb{Z}} \begin{pmatrix} A_{11}^{(n)} & A_{12}^{(n)} & A_{13}^{(n)} \\ A_{21}^{(n)} & A_{22}^{(n)} & A_{23}^{(n)} \\ A_{31}^{(n)} & A_{32}^{(n)} & A_{33}^{(n)} \end{pmatrix} \rightarrow \sum_{n \in \mathbb{Z}} \begin{pmatrix} -A_{33}^{(n)} & -A_{32}^{(n)} & -A_{31}^{(n)} \\ -A_{23}^{(n)} & -A_{22}^{(n)} & -A_{21}^{(n)} \\ -A_{13}^{(n)} & -A_{12}^{(n)} & -A_{11}^{(n)} \end{pmatrix}. \quad (34)$$

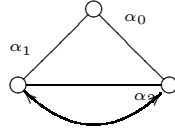
Here $K =$

$$\begin{aligned}
& \frac{1}{2} \sum_{n \in \mathbb{Z}} \begin{pmatrix} A_{11}^{(n)} - A_{33}^{(n)} & A_{12}^{(n)} - A_{32}^{(n)} & A_{13}^{(n)} - A_{31}^{(n)} \\ A_{21}^{(n)} - A_{23}^{(n)} & 0 & A_{23}^{(n)} - A_{21}^{(n)} \\ A_{31}^{(n)} - A_{13}^{(n)} & A_{32}^{(n)} - A_{12}^{(n)} & A_{33}^{(n)} - A_{11}^{(n)} \end{pmatrix} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \begin{pmatrix} (\widehat{A}_{11}^{(n)})_- & (\widehat{A}_{12}^{(n)})_- & (\widehat{A}_{13}^{(n)})_- \\ (\widehat{A}_{21}^{(n)})_- & 0 & (\widehat{A}_{23}^{(n)})_- \\ (\widehat{A}_{31}^{(n)})_- & (\widehat{A}_{32}^{(n)})_- & (\widehat{A}_{33}^{(n)})_- \end{pmatrix} \\
\text{and } P &= \frac{1}{2} \sum_{n \in \mathbb{Z}} \begin{pmatrix} A_{11}^{(n)} + A_{33}^{(n)} & A_{12}^{(n)} + A_{32}^{(n)} & A_{13}^{(n)} + A_{31}^{(n)} \\ A_{21}^{(n)} + A_{23}^{(n)} & A_{22}^{(n)} & A_{23}^{(n)} + A_{21}^{(n)} \\ A_{31}^{(n)} + A_{13}^{(n)} & A_{32}^{(n)} + A_{12}^{(n)} & A_{11}^{(n)} + A_{33}^{(n)} \end{pmatrix} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \begin{pmatrix} (\widehat{A}_{11}^{(n)})_+ & (\widehat{A}_{12}^{(n)})_+ & (\widehat{A}_{13}^{(n)})_+ \\ (\widehat{A}_{21}^{(n)})_+ & (\widehat{A}_{22}^{(n)})_+ & (\widehat{A}_{23}^{(n)})_+ \\ (\widehat{A}_{31}^{(n)})_+ & (\widehat{A}_{32}^{(n)})_+ & (\widehat{A}_{33}^{(n)})_+ \end{pmatrix} \text{ Thus } K + iP \text{ is} \\
& \frac{1}{2} \sum_{n \in \mathbb{Z}} \begin{pmatrix} (\widehat{A}_{11}^{(n)})_- + i(\widehat{A}_{11}^{(n)})_+ & (\widehat{A}_{12}^{(n)})_- + i(\widehat{A}_{12}^{(n)})_+ & (\widehat{A}_{13}^{(n)})_- + i(\widehat{A}_{13}^{(n)})_+ \\ (\widehat{A}_{21}^{(n)})_- + i(\widehat{A}_{21}^{(n)})_+ & i(\widehat{A}_{22}^{(n)})_+ & (\widehat{A}_{23}^{(n)})_- + i(\widehat{A}_{23}^{(n)})_+ \\ (\widehat{A}_{31}^{(n)})_- + i(\widehat{A}_{31}^{(n)})_+ & (\widehat{A}_{32}^{(n)})_- + i(\widehat{A}_{32}^{(n)})_+ & (\widehat{A}_{33}^{(n)})_- + i(\widehat{A}_{33}^{(n)})_+ \end{pmatrix} \quad (35)
\end{aligned}$$

such that trace of this matrix is zero. Hence the non-compact real form is $K + iP \oplus \text{Ric} \oplus \text{Rid} \in \mathfrak{sl}_{-1}^{(1)}(3, \mathbb{R})$. The Vogan diagram is

Here $K = \mathfrak{so}_{-1}^{(1)}(3)$. Thus the corresponding symmetric spaces are

$$SL_{-1}^{(1)}(3, \mathbb{R})/SO_{-1}^{(1)}(3), \quad SU^{(1)}(3)/SO_{-1}^{(1)}(3). \quad (36)$$



Thus we have completed the explicit (algebraically) calculation of affine Kac-Moody symmetric spaces associated with $A_1^{(1)}$ and $A_2^{(1)}$ and relating them with the Vogan diagrams which classify their real forms also. The next tables contain the real forms, Vogan diagrams, fixed algebras associated with the automorphism, related compact and non-compact affine kac-Moody symmetric spaces. In order to have a clear idea about the computation of fixed algebras and affine Kac-Moody symmetric spaces we have provided an appendix which contains all the necessary ingredients to understand the method of our calculations.

TABLE I: Affine Kac-Moody symmetric spaces associated with $A_{2n-1}^{(1)}$

Dynkin Diagram	Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
	$\mathfrak{su}^{(1)}(2n)$		$\mathfrak{su}^{(1)}(2n)$		
	$\mathfrak{su}_{-1}^{(1)}(p, q),$ $p + q = 2n$		$\mathfrak{su}(2n)$	$\frac{SU^{(1)}(p+q)}{SU(2n)}$	$\frac{SU_{-1}^{(1)}(p,q)}{SU(2n)}$
	$\mathfrak{su}_1^{(1)}(p, q),$ $p + q = 2n$		$\mathfrak{su}(p) \oplus \mathfrak{su}(q)$	$\frac{SU^{(1)}(p+q)}{SU(p) \oplus SU(q)}$	$\frac{SU_1^{(1)}(p,q)}{SU(p) \oplus SU(q)}$
	$\mathfrak{sl}_s^{(1)}(n, \mathbb{H})$		$\mathfrak{sp}^{(1)}(2n)$	$\frac{SU^{(1)}(2n)}{SP^{(1)}(2n)}$	$\frac{SL_s^{(1)}(n, \mathbb{H})}{SP^{(1)}(2n)}$
	$\mathfrak{sl}_{-1}^{(1)}(2n, \mathbb{R}),$ $n \geq 3$		$\mathfrak{su}^{(2)}(2n)$	$\frac{SU^{(1)}(2n)}{SU^{(2)}(2n)}$	$\frac{SL_{-1}^{(1)}(2n, \mathbb{R})}{SU^{(2)}(2n)}$
	$\mathfrak{sl}_1^{(1)}(2n, \mathbb{R}),$ $n \geq 4$		$\mathfrak{so}^{(1)}(2n)$	$\frac{SU^{(1)}(2n)}{SO^{(1)}(2n)}$	$\frac{SL_1^{(1)}(2n, \mathbb{R})}{SO^{(1)}(2n)}$
	$\mathfrak{sl}_{r,n}^{(1)}(n, \mathbb{H})$		$\mathfrak{su}^{(1)}(n)$	$\frac{SU^{(1)}(2n)}{SU^{(1)}(n)}$	$\frac{SL_{r,n}^{(1)}(n, \mathbb{H})}{SU^{(1)}(n)}$
	$\mathfrak{sl}_{r,s}^{(1)}(n, \mathbb{H})$		$\mathfrak{so}^{(2)}(2n)$	$\frac{SU^{(1)}(2n)}{SO^{(2)}(2n)}$	$\frac{SL_{r,s}^{(1)}(n, \mathbb{H})}{SO^{(2)}(2n)}$

TABLE II: Affine Kac-Moody symmetric spaces associated with $A_{2n}^{(1)}$

Dynkin Diagram	Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
	$\mathfrak{su}^{(1)}(2n+1)$		$\mathfrak{su}^{(1)}(2n+1)$		
	$\mathfrak{su}_{-1}^{(1)}(p, q),$ $p+q=2n+1$		$\mathfrak{su}(2n+1)$	$\frac{SU^{(1)}(p+q)}{SU(2n+1)}$	$\frac{SU_{-1}^{(1)}(p, q)}{SU(2n+1)}$
	$\mathfrak{su}_1^{(1)}(p, q),$ $p+q=2n+1$		$\mathfrak{su}(p) \oplus \mathfrak{su}(q)$	$\frac{SU^{(1)}(p+q)}{SU(p) \oplus SU(q)}$	$\frac{SU_1^{(1)}(p, q)}{SU(p) \oplus SU(q)}$
	$\mathfrak{sl}_{-1}^{(1)}(2n+1, \mathbb{R})$		$\mathfrak{su}^{(2)}(2n+1)$	$\frac{SU^{(1)}(2n+1)}{SU^{(2)}(2n+1)}$	$\frac{SL_{-1}^{(1)}(2n+1, \mathbb{R})}{SU^{(2)}(2n+1)}$
	$\mathfrak{sl}_1^{(1)}(2n+, \mathbb{R}),$ $n \geq 3$		$\mathfrak{so}^{(1)}(2n)$	$\frac{SU^{(1)}(2n+1)}{SO^{(1)}(2n)}$	$\frac{SL_1^{(1)}(2n+, \mathbb{R})}{SO^{(1)}(2n)}$

TABLE III: Affine Kac-Moody symmetric spaces associated with $B_n^{(1)}$

Dynkin Diagram	Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
	$\mathfrak{so}^{(1)}(2n+1)$		$\mathfrak{so}^{(1)}(2n+1)$		
	$\mathfrak{so}_{-1}^{(1)}(2, 2n-1)$		$\mathfrak{so}(2n+1)$	$\frac{SO^{(1)}(2n+1)}{SO(2n+1)}$	$\frac{SO_{-1}^{(1)}(2, 2n-1)}{SO(2n+1)}$
	$\mathfrak{so}^{(1)}(4, 2n-3)$		$\mathfrak{so}(4) \oplus \mathfrak{so}(2n-3)$	$\frac{SO^{(1)}(2n+1)}{SO(4) \oplus SO(2n-3)}$	$\frac{SO^{(1)}(4, 2n-3)}{SO(4) \oplus SO(2n-3)}$
	$\mathfrak{so}^{(1)}(6, 2n-5)$		$\mathfrak{su}^{(1)}(4) \oplus \mathfrak{so}(2n-5)$	$\frac{SO^{(1)}(2n+1)}{SU^{(1)}(4) \oplus SO(2n-5)}$	$\frac{SO^{(1)}(6, 2n-5)}{SU^{(1)}(4) \oplus SO(2n-5)}$
	$\mathfrak{so}^{(1)}(2p, 2q+1)$ $p+q=n$		$\mathfrak{so}^{(1)}(2p) \oplus \mathfrak{so}(2q+1)$	$\frac{SO^{(1)}(2n+1)}{SO^{(1)}(2p) \oplus SO(2q+1)}$	$\frac{SO^{(1)}(2p, 2q+1)}{SO^{(1)}(2p) \oplus SO(2q+1)}$
	$\mathfrak{so}^{(1)}(2n, 1)$		$\mathfrak{so}^{(1)}(2n)$	$\frac{SO^{(1)}(2n+1)}{SO^{(1)}(2n)}$	$\frac{SO^{(1)}(2n, 1)}{SO^{(1)}(2n)}$
	$\mathfrak{so}_1^{(1)}(2, 2n-1)$		$\mathfrak{so}(2n-1)$	$\frac{SO^{(1)}(2n+1)}{SO(2n-1)}$	$\frac{SO_1^{(1)}(2, 2n-1)}{SO(2n-1)}$
	$\mathfrak{so}^{(1)}(1, 2n)$		$\mathfrak{so}^{(2)}(2n)$	$\frac{SO^{(1)}(2n+1)}{SO^{(2)}(2n)}$	$\frac{SO^{(1)}(1, 2n)}{SO^{(2)}(2n)}$
	$\mathfrak{so}^{(1)}(5, 2n-4)$		$\mathfrak{su}(3) \oplus \mathfrak{so}(2n-3)$	$\frac{SO^{(1)}(2n+1)}{SU(3) \oplus SO(2n-3)}$	$\frac{SO^{(1)}(3, 2n-2)}{SU(3) \oplus SO(2n-3)}$
	$\mathfrak{so}^{(1)}(2p+1, 2q)$ $p+q=n$		$\mathfrak{so}^{(2)}(2p) \oplus \mathfrak{so}(2q+1)$	$\frac{SO^{(1)}(2n+1)}{SO^{(2)}(2p) \oplus SO(2q+1)}$	$\frac{SO^{(1)}(2p+1, 2q)}{SO^{(2)}(2p) \oplus SO(2q+1)}$
$\mathfrak{so}^{(1)}(2n-3, 4)$		$\mathfrak{so}^{(2)}(2n)$	$\frac{SO^{(1)}(2n+1)}{SO^{(2)}(2n)}$	$\frac{SO^{(1)}(2n-3, 4)}{SO^{(2)}(2n)}$	

TABLE IV: Affine Kac-Moody symmetric spaces associated with $C_{2n-1}^{(1)}$

Dynkin Diagram	Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
	$\mathfrak{sp}^{(1)}(2n-1)$		$\mathfrak{sp}^{(1)}(2n-1)$		
	$\mathfrak{sp}^{(1)}(p, q)$ $p+q=2n-1$		$\mathfrak{sp}^{(1)}(p) \oplus \mathfrak{sp}(q)$	$\frac{SP^{(1)}(p+q)}{SP^{(1)}(p) \oplus SP(q)}$	$\frac{SP^{(1)}(p, q)}{SP^{(1)}(p) \oplus SP(q)}$
	$\mathfrak{sp}_{-1}^{(1)}(2n-1, \mathbb{R})$		$\mathfrak{sp}(2n-1)$	$\frac{SP^{(1)}(2n-1)}{SP(2n-1)}$	$\frac{SP_{-1}^{(1)}(2n-1, \mathbb{R})}{SP(2n-1)}$
	$\mathfrak{sp}_1^{(1)}(2n-1, \mathbb{R})$		$\mathfrak{su}(2n-1)$	$\frac{SP^{(1)}(2n-1)}{SU(2n-1)}$	$\frac{SP_1^{(1)}(2n-1, \mathbb{R})}{SU(2n-1)}$
	$\mathfrak{sp}^{(1)}(2n-1, \mathbb{R})$		$\mathfrak{su}^{(2)}(2n-1)$	$\frac{SP^{(1)}(2n-1)}{SU^{(2)}(2n-1)}$	$\frac{SP^{(1)}(2n-1, \mathbb{R})}{SU^{(2)}(2n-1)}$

TABLE V: Affine Kac-Moody symmetric spaces associated with $C_{2n}^{(1)}$

Dynkin Diagram	Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
	$\mathfrak{sp}^{(1)}(2n)$		$\mathfrak{sp}^{(1)}(2n)$		
	$\mathfrak{sp}^{(1)}(p, q)$ $p + q = 2n, p > 1$		$\mathfrak{sp}^{(1)}(p) \oplus \mathfrak{sp}(q)$	$\frac{SP^{(1)}(p+q)}{SP^{(1)}(p) \oplus SP(q)}$	$\frac{SP^{(1)}(p, q)}{SP^{(1)}(p) \oplus SP(q)}$
	$\mathfrak{sp}_{-1}^{(1)}(2n, \mathbb{R})$		$\mathfrak{sp}(2n)$	$\frac{SP^{(1)}(2n)}{SP(2n)}$	$\frac{SP_{-1}^{(1)}(2n, \mathbb{R})}{SP(2n)}$
	$\mathfrak{sp}_1^{(1)}(2n, \mathbb{R})$		$\mathfrak{su}(2n)$	$\frac{SP^{(1)}(2n)}{SU(2n)}$	$\frac{SP_1^{(1)}(2n, \mathbb{R})}{SU(2n)}$
	$\mathfrak{sp}^{(1)}(n, \mathbb{H})$		$\mathfrak{sp}^{(1)}(n)$	$\frac{SP^{(1)}(2n)}{SP^{(1)}(n)}$	$\frac{SP^{(1)}(n, \mathbb{H})}{SP^{(1)}(n)}$
	$\mathfrak{sp}^{(1)}(2n, \mathbb{R}) \ n \geq 3$		$\mathfrak{su}^{(2)}(2n)$	$\frac{SP^{(1)}(2n)}{SU^{(2)}(2n)}$	$\frac{SP^{(1)}(2n, \mathbb{R})}{SU^{(2)}(2n)}$

TABLE VI: Affine Kac-Moody symmetric spaces associated with $D_n^{(1)}$
for even n and $n > 4$

Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
$\mathfrak{so}^{(1)}(2n)$		$\mathfrak{so}^{(1)}(2n)$		
		$\mathfrak{so}^{(1)}(2p) \oplus \mathfrak{so}^{(1)}(2q)$	$\frac{SO^{(1)}(2p+2q)}{SO^{(1)}(2p) \oplus SO(2q)}$	$\frac{SO^{(1)}(2p,2q)}{SO^{(1)}(2p) \oplus SO(2q)}$
$\mathfrak{so}^{*(1)}(2n)$		$\mathfrak{su}(n)$	$\frac{SO^{(1)}(2n)}{SU(n)}$	$\frac{SO^{*(1)}(n)}{SU(n)}$
$\mathfrak{so}_{-1}^{(1)}(2, 2n-2)$		$\mathfrak{so}(2n-2)$	$\frac{SO^{(1)}(2n)}{SO(2n-2)}$	$\frac{SO_{-1}^{(1)}(2,2n-2)}{SO(2n-2)}$
$\mathfrak{so}_{\sigma_v}^{(1)}(1, 2n-1)$		$\mathfrak{sp}^{(2)}(2n-2)$	$\frac{SO^{(1)}(2n)}{SP^{(2)}(2n-2)}$	$\frac{SO_{\sigma_v}^{(1)}(1,2n-1)}{SP^{(2)}(2n-2)}$
		$\mathfrak{so}^{(2)}(2p) \oplus \mathfrak{so}^{(2)}(2q+1)$	$\frac{SO^{(1)}(2n)}{SO^{(2)}(2p) \oplus SO(2q+1)}$	$\frac{SO_{\sigma_v}^{(1)}(2p,2q+1)}{SO^{(2)}(2p) \oplus SO(2q+1)}$
$\mathfrak{so}_{\gamma}^{(1)}(1, 2n-1)$		$\mathfrak{so}^{(1)}(2n-1)$	$\frac{SO^{(1)}(2n)}{SO^{(1)}(2n-1)}$	$\frac{SO_{\gamma}^{(1)}(1,2n-1)}{SO^{(1)}(2n-1)}$
		$\mathfrak{so}^{(1)}(2p+1) \oplus \mathfrak{so}^{(1)}(2q+1)$	$\frac{SO^{(1)}(2n)}{SO^{(1)}(2p+1) \oplus SO(2q+1)}$	$\frac{SO_{\gamma}^{(1)}(2p+1,2q+1)}{SO^{(1)}(2p+1) \oplus SO(2q+1)}$
$\mathfrak{so}_{\sigma_s}^{(1)}(1, 2n-1)$		$\mathfrak{su}^{(2)}(n)$	$\frac{SO^{(1)}(2n)}{SU^{(2)}(n)}$	$\frac{SO_{\sigma_s}^{(1)}(1,2n-1)}{SU^{(2)}(n)}$
$\mathfrak{so}_{\sigma_s}^{(1)}(n+1, n-1)$		$\mathfrak{so}^{(1)}(n)$	$\frac{SO^{(1)}(2n)}{SO^{(1)}(n)}$	$\frac{SO_{\sigma_s}^{(1)}(n+1,n-1)}{SO^{(1)}(n)}$

TABLE VII: Affine Kac-Moody symmetric spaces associated with $D_n^{(1)}$
for n odd and $n > 4$

Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
$\mathfrak{so}^{(1)}(2n)$		$\mathfrak{so}^{(1)}(2n)$		
		$\mathfrak{so}^{(1)}(2p) \oplus \mathfrak{so}^{(1)}(2q)$	$\frac{SO^{(1)}(2p+2q)}{SO^{(1)}(2p) \oplus SO(2q)}$	$\frac{SO^{(1)}(2p, 2q)}{SO^{(1)}(2p) \oplus SO(2q)}$
$\mathfrak{so}^{*(1)}(2n)$		$\mathfrak{su}(n)$	$\frac{SO^{(1)}(2n)}{SU(n)}$	$\frac{SO^{*(1)}(n)}{SU(n)}$
$\mathfrak{so}_{-1}^{(1)}(2, 2n-2)$		$\mathfrak{so}(2n-2)$	$\frac{SO^{(1)}(2n)}{SO(2n-2)}$	$\frac{SO_{-1}^{(1)}(2, 2n-2)}{SO(2n-2)}$
$\mathfrak{so}_{\sigma_v}^{(1)}(1, 2n-1)$		$\mathfrak{sp}^{(2)}(2n-2)$	$\frac{SO^{(1)}(2n)}{SP^{(2)}(2n-2)}$	$\frac{SO_{\sigma_n}^{(1)}(1, 2n-1)}{SP^{(2)}(2n-2)}$
		$\mathfrak{so}^{(2)}(2p) \oplus \mathfrak{so}^{(2)}(2q+1)$	$\frac{SO^{(1)}(2n)}{SO^{(2)}(2p) \oplus SO(2q+1)}$	$\frac{SO^{(1)}(2p, 2q+1)}{SO^{(2)}(2p) \oplus SO(2q+1)}$
$\mathfrak{so}_{\gamma}^{(1)}(1, 2n-1)$		$\mathfrak{so}^{(1)}(2n-1)$	$\frac{SO^{(1)}(2n)}{SO^{(1)}(2n-1)}$	$\frac{SO_{\gamma}^{(1)}(1, 2n-1)}{SO^{(1)}(2n-1)}$
		$\mathfrak{so}^{(1)}(2p+1) \oplus \mathfrak{so}^{(1)}(2q+1)$	$\frac{SO^{(1)}(2n)}{SO^{(1)}(2p+1) \oplus SO(2q+1)}$	$\frac{SO^{(1)}(2p+1, 2q+1)}{SO^{(1)}(2p+1) \oplus SO(2q+1)}$
$\mathfrak{so}_{\sigma_s}^{(1)}(1, 2n-1)$		$\mathfrak{so}^{(2)}(n-1)$	$\frac{SO_{\sigma_s}^{(1)}(1, 2n-1)}{SO^{(2)}(n-1)}$	$\frac{SO_{\sigma_s}^{(1)}(1, 2n-1)}{SO^{(2)}(n-1)}$

IV. APPENDIX

A. Classical Irreducible Reduced Root systems

In this subsection we have given the irreducible reduced root system of complex semi simple Lie algebras A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$ and D_n for $n \geq 4$. In case of A_n the under lying vector space $V = \{v \in \mathbb{R}^{n+1} \mid \langle v, e_1 + \dots + e_{n+1} \rangle = 0\}$ and for rest algebras $V = \mathbb{R}^n \cdot \Delta$ denotes the root system which is a subspace of some $\mathbb{R}^k = \sum_{i=1}^k a_i e_i$. Here $\{e_i\}$ is the standard orthonormal basis and a_i 's are real. Δ^+ is the positive root system and Π is simple root system.

TABLE VIII:

g	Δ	Δ^+	Π	Largest Root
$A_n = \mathfrak{sl}(n+1, \mathbb{C})$	$\{e_i - e_j \mid i \neq j\}$	$\{e_i - e_j \mid i < j\}$	$\{e_1 - e_2, \dots, e_n - e_{n+1}\}$	$e_1 - e_{n+1}$
$B_n = \mathfrak{so}(2n+1, \mathbb{C})$	$\{\pm e_i \pm e_j \mid i < j\} \cup \{\pm e_i\}$	$\{e_i \pm e_j \mid i < j\} \cup \{e_i\}$	$\{e_1 - e_2, \dots, e_{n-1} - e_n, e_n\}$	$e_1 + e_2$
$C_n = \mathfrak{sp}(n, \mathbb{C})$	$\{\pm e_i \pm e_j \mid i < j\} \cup \{\pm 2e_i\}$	$\{e_i \pm e_j \mid i < j\} \cup \{2e_i\}$	$\{e_1 - e_2, \dots, e_{n-1} - e_n, 2e_n\}$	$2e_1$
$D_n = \mathfrak{so}(2n, \mathbb{C})$	$\{\pm e_i \pm e_j \mid i < j\}$	$\{e_i \pm e_j \mid i < j\}$	$\{e_1 - e_2, \dots, e_{n-1} - e_n, e_{n-1} + e_n\}$	$e_1 + e_2$

B. Diagram Automorphism

The diagram automorphism for affine Lie algebra are as follows. For $A_n^{(1)}$, the automorphism group of the Dynkin diagram is the dihedral group D_{n+1} which is generated by reflection $s : i \mapsto n+1-i \pmod{n+1}$ and the rotation $r : i \mapsto i+1 \pmod{n+1}$ which is of order $r+1$. For $D_r^{(1)}$, the automorphism group is generated by the vector automorphism σ_v , the spinor automorphism σ_s and the conjugation γ . σ_v acts as $0 \leftrightarrow 1$, $r \leftrightarrow r-1$ and $i \mapsto i$ else, and hence is of order 2. The map γ acts as $r \leftrightarrow r-1$ and $i \mapsto i$ else. If r is even, σ_s acts as $i \mapsto r-i$ is of order 2 while for odd r the prescription $i \mapsto r-i$ only holds for $2 \leq i \leq r-2$ and is supplemented by $0 \mapsto r \mapsto 1 \mapsto r-1 \mapsto 0$. For the untwisted algebra $g = B_r^{(1)}, C_r^{(1)}$ and for twisted algebra $g = B_r^{(2)}, C_r^{(2)}$, there is only a single non-trivial automorphism γ which is a reflection [11].

C. Classical Non-compact real affine Kac-Moody Lie Algebras

Let $g_{\mathbb{R}}$ be a real affine Kac-Moody Lie algebra and \mathfrak{t}_0 be the fixed subalgebra of a Cartan involution on $g_{\mathbb{R}}$. \mathfrak{c}_0 is the center of \mathfrak{t}_0 , then simple roots of \mathfrak{t}_0 are obtained as follows. When the automorphism in the Vogan diagram is non trivial, we know that \mathfrak{t}_0 is semisimple. The simple roots for \mathfrak{t}_0 then include the compact imaginary simple roots and the average of the members of each 2-element orbit of simple roots. If the Vogan diagram has no painted imaginary root, there is no other simple root for \mathfrak{t}_0 . Otherwise there is one other simple root for \mathfrak{t}_0 , obtained by taking a minimal complex root containing the painted imaginary root in its expansion and averaging it over its 2-element orbit under the automorphism. When the automorphism is trivial, either $\dim \mathfrak{c}_0 = 1$, in this case the simple roots for \mathfrak{t}_0 are the compact simple roots for \mathfrak{g}_0 , or else $\dim \mathfrak{c}_0 = 0$, in this case the simple roots for \mathfrak{t}_0 are the compact imaginary simple roots for \mathfrak{g}_0 and one other compact imaginary root. In latter case this other compact imaginary root is the unique smallest root containing the non-compact simple root twice in its expansion. We have discussed below in detail, for each real algebra separately.

- $\mathfrak{sl}_s^{(1)}(n, \mathbb{H}), n \text{ even} \geq 2$

Vogan diagram:

A_{n-1} , non trivial automorphism,
no imaginary simple roots

$\mathfrak{t}_0 = \mathfrak{sp}^{(1)}(n)$

Simple roots for \mathfrak{t}_0 :

$e_{2n} - e_1, e_n - e_{n+1}$ and
all $\frac{1}{2}(e_i - e_{i+1} + e_{2n-i} - e_{2n+1-i})$ for $1 \leq i \leq (n-1)$

- $\mathfrak{sl}_{-1}^{(1)}(2n, \mathbb{R}), n \geq 3$
 Vogan diagram:
 A_{n-1} , non trivial automorphism,
 unique imaginary simple root $e_{2n} - e_1$
 $\mathfrak{t}_\circ = \mathfrak{su}^{(2)}(2n)$
 Simple roots for \mathfrak{t}_\circ :
 $\frac{1}{2}(e_{n-1} + e_n - e_{n+1} - e_{n+2})$ and
 all $\frac{1}{2}(e_i - e_{i+1} + e_{2n-i} - e_{2n+1-i})$ for $1 \leq i \leq (n-1)$
- $\mathfrak{sl}_{-1}^{(1)}(2n+1, \mathbb{R}), n \geq 3$
 Vogan diagram:
 A_{n-1} , non trivial automorphism,
 no imaginary simple roots
 $\mathfrak{t}_\circ = \mathfrak{su}^{(2)}(2n+1)$
 Simple roots for \mathfrak{t}_\circ :
 $e_{2n+1} - e_1, \frac{1}{2}(e_n - e_{n+2})$ and
 all $\frac{1}{2}(e_i - e_{i+1} + e_{2n+1-i} - e_{2n+2-i})$ for $1 \leq i \leq (n-1)$
- $\mathfrak{sl}_1^{(1)}(2n, \mathbb{R}), n \geq 4$
 Vogan diagram:
 A_{n-1} , non trivial automorphism,
 two imaginary simple roots $e_{2n} - e_1, e_n - e_{n+1}$
 $\mathfrak{t}_\circ = \mathfrak{so}^{(1)}(2n)$
 Simple roots for \mathfrak{t}_\circ :
 $\frac{1}{2}(e_{n-1} + e_n - e_{n+1} - e_{n+2}), \frac{1}{2}(e_{2n-1} + e_{2n} - e_1 - e_2)$ and
 all $\frac{1}{2}(e_i - e_{i+1} + e_{2n+1-i} - e_{2n+2-i})$ for $1 \leq i \leq (n-1)$
- $\mathfrak{sl}_1^{(1)}(2n+1, \mathbb{R}), n \geq 4$
 Vogan diagram:
 A_{n-1} , non trivial automorphism,
 one imaginary simple roots $e_{2n+1} - e_1$
 $\mathfrak{t}_\circ = \mathfrak{so}^{(1)}(2n)$
 Simple roots for \mathfrak{t}_\circ :
 $\frac{1}{2}(e_n - e_{n+2}), \frac{1}{2}(e_{2n} + e_{2n+1} - e_1 - e_2)$ and
 all $\frac{1}{2}(e_i - e_{i+1} + e_{2n+1-i} - e_{2n+2-i})$ for $1 \leq i \leq (n-1)$
- $\mathfrak{sl}_{r,n}^{(1)}(2n, \mathbb{H}), n \geq 4$
 Vogan diagram:
 A_{n-1} , non trivial automorphism,
 no imaginary simple roots
 $\mathfrak{t}_\circ = \mathfrak{su}^{(1)}(n)$
 Simple roots for \mathfrak{t}_\circ :
 $\frac{1}{2}(e_n - e_{n+1} + e_{2n} - e_1)$ and
 all $\frac{1}{2}(e_i - e_{i+1} + e_{n+i} - e_{n+1-i})$ for $1 \leq i \leq (n-1)$
- $\mathfrak{sl}_{rs}^{(1)}(n, \mathbb{H}), n \geq 4$
 Vogan diagram:
 A_{n-1} , non trivial automorphism,
 no imaginary simple roots
 $\mathfrak{t}_\circ = \mathfrak{so}^{(2)}(2n)$
 Simple roots for \mathfrak{t}_\circ :
 $\frac{1}{2}(e_{2n} - e_2), \frac{1}{2}(e_n - e_{n+2})$ and
 all $\frac{1}{2}(e_{i+1} - e_{i+2} + e_{2n-i} - e_{2n+1-i})$ for $1 \leq i \leq (n-2)$
- $\mathfrak{su}_{-1}^{(1)}(p, q), p + q = 2n$
 Vogan diagram:
 A_{2n-1} , trivial automorphism,
 unique imaginary simple root $e_{2n} - e_1$

$$\mathfrak{t}_\circ = \mathfrak{su}(2n)$$

Simple roots for \mathfrak{t}_\circ :

compact simple roots only

- $\mathfrak{su}_1^{(1)}(p, q), p + q = 2n$

Vogan diagram:

A_{2n-1} , trivial automorphism,
two imaginary simple roots $e_{2n} - e_1, e_p - e_{p+1}$

$$\mathfrak{t}_\circ = \mathfrak{su}(p) \oplus \mathfrak{su}(q)$$

Simple roots for \mathfrak{t}_\circ :

compact simple roots only

- $\mathfrak{so}_{-1}^{(1)}(2, 2n - 1)$

Vogan diagram:

B_n , trivial automorphism,
one imaginary simple root $e_1 - e_2$

$$\mathfrak{t}_\circ = \mathfrak{so}(2n + 3)$$

Simple roots for \mathfrak{t}_\circ :

compact simple roots only

- $\mathfrak{so}^{(1)}(2p, 2q + 1), p + q = n$

Vogan diagram:

B_n , trivial automorphism,
one imaginary simple root $e_p - e_{p+1}$

$$\mathfrak{t}_\circ = \begin{cases} \mathfrak{so}(4) \oplus \mathfrak{so}(2n - 3), & \text{if } p = 2 \\ \mathfrak{su}^{(1)}(4) \oplus \mathfrak{so}(2n - 5), & \text{if } p = 3 \\ \mathfrak{so}^{(1)}(2n) & \text{if } q = 0 \\ \mathfrak{so}^{(1)}(2p) \oplus \mathfrak{so}(2q), & \text{if else} \end{cases}$$

Simple roots for \mathfrak{t}_\circ :

compact simple roots and
 $\begin{cases} e_{p-1} + e_p & \text{when } p \geq 3 \\ \text{no other} & \text{when } p = 2 \end{cases}$

- $\mathfrak{so}^{(1)}(1, 2n)$

Vogan diagram:

B_n , non-trivial automorphism,
no imaginary simple roots are painted

$$\mathfrak{t}_\circ = \mathfrak{so}^{(2)}(2n)$$

Simple roots for \mathfrak{t}_\circ :

compact simple roots only and $-e_2$

- $\mathfrak{so}_1^{(1)}(2, 2n - 1)$

Vogan diagram:

B_n , trivial automorphism,
two imaginary simple roots are painted $e_1 - e_2, -e_1 - e_2$

$$\mathfrak{t}_\circ = \mathfrak{so}(2n + 1)$$

Simple roots for \mathfrak{t}_\circ :

compact simple roots only

- $\mathfrak{so}^{(1)}(2p + 1, 2q), p + q = n$

Vogan diagram:

B_n , non-trivial automorphism,
one imaginary simple root $e_p - e_{p+1}$

$$\mathfrak{t}_\circ = \begin{cases} \mathfrak{su}(3) \oplus \mathfrak{so}(2n - 3), & \text{if } p = 2 \\ \mathfrak{so}^{(2)}(2n), & \text{if } q = 2 \\ \mathfrak{so}^{(2)}(2p) \oplus \mathfrak{so}(2q + 1), & \text{if else} \end{cases}$$

Simple roots for \mathfrak{t}_\circ :

compact simple roots and $-e_2, e_p$

- $\mathfrak{sp}^{(1)}(p, q)$, $p + q = 2n$ or $p + q = 2n - 1$
 Vogan diagram:
 C_n , trivial automorphism,
 p^{th} simple root painted $e_p - e_{p+1}$,
 $\mathfrak{t}_\circ = \begin{cases} \mathfrak{su}^{(1)}(2) \oplus \mathfrak{sp}(q) & \text{if } p = 1, \forall q \\ \mathfrak{sp}^{(1)}(p) \oplus \mathfrak{sp}(q) & \text{if } p > 1, \forall q \end{cases}$
 Simple roots for \mathfrak{t}_\circ : compact simple roots and $2e_p$
- $\mathfrak{sp}_{-1}^{(1)}(n, \mathbb{R})$
 Vogan diagram:
 C_n , trivial automorphism,
 n^{th} simple root painted $2e_n$,
 $\mathfrak{t}_\circ = \mathfrak{sp}(n)$
 Simple roots for \mathfrak{t}_\circ : compact simple roots only
- $\mathfrak{sp}_1^{(1)}(n, \mathbb{R})$
 Vogan diagram:
 C_n , trivial automorphism,
 n^{th} and affine simple roots are painted $2e_n, -2e_1$
 $\mathfrak{t}_\circ = \mathfrak{su}(n)$
 Simple roots for \mathfrak{t}_\circ : compact simple roots only
- $\mathfrak{sp}^{(1)}(2n - 1, \mathbb{R})$
 Vogan diagram:
 C_n , non-trivial automorphism,
 no imaginary simple root
 $\mathfrak{t}_\circ = \mathfrak{su}^{(2)}(2n)$
 Simple roots for \mathfrak{t}_\circ :
 $(e_{2n-1} - e_1)$ and
 all $\frac{1}{2}(e_i - e_{i+1} + e_{2n-1-i} - e_{2n-i})$ for $1 \leq i \leq (n - 1)$
- $\mathfrak{sp}^{(1)}(n, \mathbb{H})$
 Vogan diagram:
 C_n , non-trivial automorphism,
 no imaginary simple root
 $\mathfrak{t}_\circ = \mathfrak{su}^{(2)}(2n)$
 Simple roots for \mathfrak{t}_\circ :
 $(e_n - e_{n+1}), (e_{2n} - e_1)$ and
 all $\frac{1}{2}(e_i - e_{i+1} + e_{2n-i} - e_{2n+1-i})$ for $1 \leq i \leq (n - 1)$
- $\mathfrak{sp}^{(1)}(2n, \mathbb{R})$
 Vogan diagram:
 C_n , non-trivial automorphism,
 unique imaginary simple root $e_n - e_{n+1}$ painted
 $\mathfrak{t}_\circ = \mathfrak{su}^{(2)}(2n)$
 Simple roots for \mathfrak{t}_\circ :
 $\frac{1}{2}(e_{n-1} + e_n - e_{n+1} - e_{n+2})$ and
 all $\frac{1}{2}(e_i - e_{i+1} + e_{2n-i} - e_{2n+1-i})$ for $1 \leq i \leq (n - 1)$
- $\mathfrak{so}^{(1)}(2p, 2q)$, $p + q = n$
 Vogan diagram:
 D_{p+q} , trivial automorphism,
 p^{th} simple root painted $e_p - e_{p+1}$,
 $\mathfrak{t}_\circ = \begin{cases} \mathfrak{su}^{(1)}(4) \oplus \mathfrak{so}(2n - 6), & \text{if } p = 3 \\ \mathfrak{so}^{(1)}(2p) \oplus \mathfrak{so}(2q), & \text{if } p > 3 \end{cases}$
 Simple roots for \mathfrak{t}_\circ :

compact simple roots and

$$\begin{cases} e_{p-1} + e_p & \text{when } p \geq 2 \\ \text{no other} & \text{when } p = 1 \end{cases}$$

- $\mathfrak{so}^{*(1)}(2n)$
 Vogan diagram:
 D_n , trivial automorphism,
 two imaginary simple roots $(e_1 - e_2), (e_{n-1} - e_n)$ painted
 $\mathfrak{t}_\circ = \mathfrak{su}(n)$
 Simple roots for \mathfrak{t}_\circ : compact simple roots only
- $\mathfrak{so}_{-1}^{(1)}(2, 2n - 2)$
 Vogan diagram:
 D_n , trivial automorphism,
 two imaginary simple roots $(e_1 - e_2), -(e_1 - e_2)$ painted
 $\mathfrak{t}_\circ = \mathfrak{su}(2n - 2)$
 Simple roots for \mathfrak{t}_\circ : compact simple roots only
- $\mathfrak{so}_{\sigma_v}^{(1)}(1, 2n - 1)$
 Vogan diagram:
 D_n , non-trivial automorphism,
 no imaginary simple root painted
 $\mathfrak{t}_\circ = \mathfrak{so}^{(2)}(2n - 2)$
 Simple roots for \mathfrak{t}_\circ : $-e_2, e_{n-1}$ and all $(e_i - e_{i+1})$ for $2 \leq i \leq (n - 2)$
- $\mathfrak{so}_{\sigma_v}^{(1)}(2p + 1, 2q + 1), p + q = n - 1$
 Vogan diagram:
 D_n , non-trivial automorphism,
 p^{th} simple root painted $e_p - e_{p+1}$,
 $\mathfrak{t}_\circ = \begin{cases} \mathfrak{su}(3) \oplus \mathfrak{so}(2n - 5), & \text{if } p = 2 \\ \mathfrak{so}^{(2)}(2p) \oplus \mathfrak{so}^{(1)}(2q + 1), & \text{if } p \geq 3 \end{cases}$
 Simple roots for \mathfrak{t}_\circ :
 $-e_2, e_{n-1}, e_p$ and
 $e_i - e_{i+1}$ for $2 \leq i \leq p - 1$ and $p + 1 \leq i \leq n - 2$
- $\mathfrak{so}_{\gamma}^{(1)}(1, 2n - 1)$
 Vogan diagram:
 D_n , non-trivial automorphism,
 no imaginary simple root painted
 $\mathfrak{t}_\circ = \mathfrak{so}^{(1)}(2n - 1)$
 Simple roots for \mathfrak{t}_\circ :
 $(e_1 - e_2), -(e_1 + e_2), e_{n-1}$ and
 all $(e_i - e_{i+1})$ for $2 \leq i \leq (n - 2)$
- $\mathfrak{so}_{\gamma}^{(1)}(2p + 1, 2q + 1), p + q = n - 1$
 Vogan diagram:
 D_n , non-trivial automorphism,
 p^{th} simple root painted $e_p - e_{p+1}$,
 $\mathfrak{t}_\circ = \begin{cases} \mathfrak{so}^{(2)}(2n), & \text{if } p = 1 \\ \mathfrak{sp}^{(1)}(2) \oplus \mathfrak{so}(2q + 1), & \text{if } p = 2 \\ \mathfrak{so}^{(1)}(2p + 1) \oplus \mathfrak{so}(2q + 1), & \text{if } p \geq 3 \end{cases}$
 Simple roots for \mathfrak{t}_\circ :
 For $p = 1$ $e_p, -(e_1 + e_2), e_{n-1}$,
 all $(e_i - e_{i+1})$ for $p + 1 \leq i \leq (n - 2)$
 For $p \neq 1$ $(e_1 - e_2), -(e_1 + e_2), e_{n-1}, e_p$ and
 all $(e_i - e_{i+1})$ for $2 \leq i \leq (p - 1)$ and $p + 1 \leq i \leq (n - 2)$

- $\mathfrak{so}_{\sigma_s}^{(1)}(1, 2n-1)$, n even

Vogan diagram:

D_n , non-trivial automorphism,
no imaginary simple root painted

$$\mathfrak{t}_\circ = \mathfrak{su}^{(2)}(n)$$

Simple roots for \mathfrak{t}_\circ :

$\frac{1}{2}(e_1 - e_2 + e_{n-1} - e_n)$, $\frac{1}{2}(-e_1 - e_2 + e_{n-1} + e_n)$, $(e_{\frac{n}{2}} - e_{\frac{n}{2}+1})$ and
all $\frac{1}{2}(e_i - e_{i+1} + e_{n-i} - e_{n+1-i})$ for $2 \leq i \leq (n-2)$

- $\mathfrak{so}_{\sigma_s}^{(1)}(1, 2n-1)$, n even

Vogan diagram:

D_n , non-trivial automorphism,
unique imaginary simple root $(e_{\frac{n}{2}} - e_{\frac{n}{2}+1})$ painted

$$\mathfrak{t}_\circ = \mathfrak{so}^{(1)}(n)$$

Simple roots for \mathfrak{t}_\circ :

$\frac{1}{2}(e_1 - e_2 + e_{n-1} - e_n)$, $\frac{1}{2}(-e_1 - e_2 + e_{n-1} + e_n)$,
 $\frac{1}{2}(e_{\frac{n}{2}-1} + e_{\frac{n}{2}} - e_{\frac{n}{2}+1} - e_{\frac{n}{2}+2})$ and
all $\frac{1}{2}(e_i - e_{i+1} + e_{n-i} - e_{n+1-i})$ for $2 \leq i \leq (n-2)$

- $\mathfrak{so}_{\sigma_s}^{(1)}(1, 2n-1)$, n odd

Vogan diagram:

D_n , non-trivial automorphism,
no imaginary simple root painted

$$\mathfrak{t}_\circ = \mathfrak{so}^{(1)}(n)$$

Simple roots for \mathfrak{t}_\circ :

$\frac{1}{2}(e_1 - e_2 + e_{n-1} - e_n)$, $\frac{1}{2}(-e_1 - e_2 + e_{n-1} + e_n)$, and
all $\frac{1}{2}(e_i - e_{i+1} + e_{n-i} - e_{n+1-i})$ for $2 \leq i \leq (n-2)$

D. Notations used

- $gl(n, \mathbb{C})$, $(gl(n, \mathbb{R}))$: {all $n \times n$ complex (real matrices)}
- $sl(n, \mathbb{C})$, $(sl(n, \mathbb{R}))$: {all $n \times n$ complex (real matrices) of trace zero}
- $sl_1^{(1)}(n, \mathbb{R})$: $sl(n, \mathbb{R}) \otimes \mathbb{C}[t, t^{-1}] \oplus Ric \oplus Rid$ with $u = 1$
- $sl_{-1}^{(1)}(n, \mathbb{R})$: $sl(n, \mathbb{R}) \otimes \mathbb{C}[t, t^{-1}] \oplus Ric \oplus Rid$ with $u = -1$
- $so(n)$: $\{X \in gl(n, \mathbb{R}) \mid X + X^* = 0\}$
- $so_1^{(1)}(n)$: $so(n) \otimes \mathbb{C}[t, t^{-1}] \oplus Ric \oplus Rid$ with $u = 1$
- $so_{-1}^{(1)}(n)$: $so(n) \otimes \mathbb{C}[t, t^{-1}] \oplus Ric \oplus Rid$ with $u = -1$
- $sp(n)$: $\{X \in gl(n, \mathbb{H}) \mid X + X^* = 0\}$
- $su(n)$: $\{X \in gl(n, \mathbb{C}) \mid X + X^* = 0 \text{ and } TrX = 0\}$
- $su^{(1)}(n)$: $\{X \in su(n) \otimes \mathbb{C}[t, t^{-1}] \oplus Ric \oplus Rid \mid X + X^* = 0 \text{ and } TrX = 0\}$
- $su^{(2)}(n)$: $\sum_{p=0}^1 \sum_{j \text{ mod } 2=p} t^j \otimes su(n)_p^{(2)} \oplus Ric \oplus Rid$ where $su(n)_0^{(2)}$ and $su(n)_1^{(2)}$ are the eigenspaces corresponding to eigenvalues 1 and $e^{\pi i}$ respectively. That is $a \in su(n)_p^{(2)}$ if $a \in su(n)$ and $\Psi_\tau(a) = e^{\pi i p}$ where Ψ_τ is the outer automorphism of $su(n)$.
- $u(p, q)$: $\left\{ \begin{pmatrix} Z_1 & Z_2 \\ \overline{Z_2} & Z_3 \end{pmatrix} \mid \begin{array}{l} Z_1, Z_3 \text{ skew Hermitian of order } p \text{ and } q \\ \text{respectively, } Z_2 \text{ is arbitrary} \end{array} \right\}$

- $su(p, q) : \left\{ \begin{pmatrix} Z_1 & Z_2 \\ \overline{Z_2}^t & Z_3 \end{pmatrix} \mid \begin{array}{l} Z_1 \text{ skew Hermitian of order } p \\ Z_3 \text{ skew Hermitian of order } q \\ TrZ_1 + TrZ_3 = 0, Z_2 \text{ is arbitrary} \end{array} \right\}$
- $su_1^{(1)}(p, q) : \left\{ \begin{pmatrix} X_1 & X_2 \\ \overline{X_2}^t & X_3 \end{pmatrix} \mid \begin{array}{l} X_i = Z_i \otimes \mathbb{C}[t, t^{-1}] \\ X_1 \text{ skew Hermitian of order } p \\ X_3 \text{ skew Hermitian of order } q \\ TrX_1 + TrX_3 = 0, X_2 \text{ is arbitrary} \end{array} \right\}$
- $su_{-1}^{(1)}(p, q) : \left\{ \begin{pmatrix} X_1 & X_2 \\ -\overline{X_2}^t & X_3 \end{pmatrix} \mid \begin{array}{l} X_i = Z_i \otimes \mathbb{C}[t, t^{-1}] \\ X_1 \text{ skew Hermitian of order } p \\ X_3 \text{ skew Hermitian of order } q \\ TrX_1 + TrX_3 = 0, X_2 \text{ is arbitrary} \end{array} \right\}$
- $su^*(2n) : \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\overline{Z_2} & \overline{Z_1} \end{pmatrix} \mid \begin{array}{l} Z_1, Z_2 \text{ } n \times n \text{ complex matrix} \\ TrZ_1 + Tr\overline{Z_1} = 0 \end{array} \right\}$
- $sp(n, \mathbb{C}) : \left\{ \begin{pmatrix} Z_1 & Z_2 \\ \overline{Z_3} & -Z_1^t \end{pmatrix} \mid \begin{array}{l} Z_i \text{ } n \times n \text{ complex matrix} \\ Z_2, Z_3 \text{ are symmetric} \end{array} \right\}$

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