On fuzzy weakly semi-continuous functions

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Abstract

This paper is devoted to the introduction and study of fuzzy weakly semi-continuous functions between fuzzy topological spaces. Some properties of these functions are characterized in terms of quasi coincidence, quasi neighborhoods, \(\theta\)-neighborhoods etc. Furthermore, some relations connecting fuzzy weakly semi-continuous functions and fuzzy retractions have been established.

Key words: Fuzzy weakly semi-continuous function; Quasi coincidence; Quasi-neighborhood; \(\theta\)-neighborhood; Semi-open set; Semi-closed set; Regular open set; Fuzzy retract

1. Introduction

The concept of fuzzy set was introduced by Zadeh in his classic paper [16]. Azad [2] has introduced the concepts of fuzzy semi-open and semi-closed sets. Since then, many authors including Azad have used these concepts to define and study fuzzy semi-continuous, semi-open and semi-closed mappings between fuzzy topological spaces (fts). Noteworthy among them are Mukherjee and Sinha [7–9], El-Monsef and Ghanim [4] and Ghose [5]. The concept of almost continuity, strong continuity and principal super-connectedness for fts have been, respectively, discussed by Nanda [10–12]. In this note we introduce fuzzy weakly semi-continuous functions which are natural generalization of fuzzy semi-continuous functions. We have brought out characterization of such functions and also have investigated some of their properties in Section 4 by using the notions of quasi-coincidence (q-coincidence), quasi-neighborhood (q-nbd), and \(\theta\)-cluster points as introduced by Ming and Ming [13] and Kotze [6]. This concept of q-coincidence is found to be well suited to the fuzzy situations. In Section 5, we have investigated some relations connecting fuzzy weakly semi-continuous functions and fuzzy retractions.

2. Preliminaries

The definitions and results not explained in this paper are already standardized by now and can be found in [1–3, 10, 14–16]. But for the sake of completeness, we recall some of these definitions

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and results. So far as the notations are concerned, we shall write $\tilde{A}$, $A^0$, $\bar{A}$, $A'$ and $A^i$ to mean, respectively, the fuzzy closure, fuzzy interior, fuzzy semi-closure, fuzzy semi-interior and fuzzy complement of a fuzzy set $A$. The words ‘fuzzy’, ‘neighborhood’ and ‘fuzzy topological space’ will be abbreviated as ‘f’, ‘nbd’ and ‘fts’, respectively. FSO($X$) and $I^X$ will, respectively, mean the set of fuzzy semi-open sets and the set of all fuzzy sets on a nonempty set $X$.

**Definition 2.1** [2]. Let $A$ be a fuzzy set in an fts $(X, T)$. Then $A$ is called
(a) f-semi-open if there is an f-open set $B$ in $(X, T)$ such that $B \subseteq A \subseteq \overline{B}$,
(b) f-semi-closed if there is an f-closed set $B$ in $(X, T)$ such that $B^0 \subseteq A \subseteq B$,
(c) f-regular open if $\overline{A}^0 = A$,
(d) f-regular closed if $\overline{A}^0 = A$.

**Definition 2.2** [3]. A mapping $f:(X, T_1) \rightarrow (Y, T_2)$ is said to be
(a) fuzzy continuous (f-continuous) if $f^{-1}(B) \in T_1$ for all $B \in T_2$,
(b) fuzzy open (f-open) if $f(A) \in T_2$ for all $A \in T_1$,
(c) fuzzy closed (f-closed) if $f(A')$ is fuzzy closed for all $A \in T_1$.

**Definition 2.3** [4, 2]. A mapping $f:X \rightarrow Y$ from an fts $X$ to an fts $Y$ is said to be
(a) f-semi-continuous if for each f-point $x_0$ in $X$ and any f-open set $B$ in $Y$ with $f(x_0) \in B$, there exists $A \in FSO(X)$ such that $x_0 \in A$ and $f(A) <\sim B$.
(b) A fuzzy point $x_0$ is said to be contained in a fuzzy set $A$ or $x_0$ belongs to $A$, i.e. $x_0 \in A$ if $\alpha \leq A(x)$.

**Theorem 2.5** [13]. Let $A \in I^X$. Then $A$ is the union of all its fuzzy points, i.e., $A = \bigvee_{x_0 \in A} x_0$.

**Theorem 2.6** [3]. Let $f$ be a function from an fts $X$ to an fts $Y$. Then the following statements hold:
1. If $A \leq B$, then $f(A) \leq f(B)$ for all $A, B \in I^X$.
2. If $C \leq D$, then $f^{-1}(C) \leq f^{-1}(D)$ for all $C, D \in I^Y$.
3. For any $A \in I^X$, $A \leq f^{-1}(f(A))$.
4. For any $B \in I^Y$, $f(A') \leq B$.
5. For any $B \in I^Y$, $f^{-1}(B^c) = (f^{-1}(B))^c$.
6. For any $A \in I^X$, $f(A') = (f(A))^c$.

**Theorem 2.7** [5]. A mapping $f:X \rightarrow Y$ from an fts $X$ to an fts $Y$ is f-semi-continuous if for any f-point $x_0$ in $X$ and any f-open set $B$ in $Y$ with $f(x_0) \in B$, there exists $A \in FSO(X)$ such that $x_0 \in A$ and $f(A) \leq B$.

**Definition 2.8** [13]. An f-set $A$ is said to be quasi-coincident (q-coincident) with an f-set $B$, if there exists at least one point $x \in X$ such that $A(x) + B(x) > 1$. It is denoted by $A \preceq B$. $A \prec B$ means that $A$ and $B$ are not q-coincident.

For two fuzzy sets $A$ and $B$, $A \leq B$ iff $A(x) \leq B(x)$ for each $x \in X$. Note that $A \leq B$ iff $A \preceq B$.

**Definition 2.9** [5]. (1) An f-set $A$ is called an f-q-nbd (f-semi-q-nbd) of an f-point, $x_0$ in an fts $(X, T)$ iff there exists an f-open set (f-semi-open set) $B$ in $X$ such that $x_0 \preceq B$.

(2) An f-point $x_0$ in an fts $(X, T)$ is called an f-semi-cluster point of an f-set $A$ iff every f-semi-q-nbd of $x_0$ is q-coincident with $A$. The set of all f-semi-cluster points of an f-set $A$ is called f-semi-closure of $A$ and denoted by $\tilde{A}$. It is proved in [5] that $\tilde{A} = \bigvee \{V: V \preceq A, \text{ where } V \text{ is f-semi-closed in } X\}$. $A$ is f-semi-closed iff $A = \tilde{A}$.

(3) The f-semi-interior of an f-set $A$, denoted by $A^i$, is defined as $A^i = \bigvee \{U: U \leq A, U \in FSO(X)\}$.

**Theorem 2.10** [4]. For any f-set $A$ in an fts $(X, T)$, $A^0 \leq A^i \leq A \leq \tilde{A} \leq \tilde{A}$.

**Theorem 2.11** [7]. If $f:X \rightarrow Y$ is f-semi-continuous and almost f-open then $f$ is f-irresolute.
Definition 2.12 [8]. An fts \((X, T)\) is fuzzy regular iff for each f-point \(x_a\) in \(X\) and each f-open q-nbd \(U\) of \(x_a\), there exists an f-open q-nbd \(V\) of \(x_a\) such that \(V \subseteq U\).

Definition 2.13 [8]. (1) An f-set \(A\) in an fts \((X, T)\) is said to be an f-\(\theta\)-nbd (f-semi-\(\theta\)-nbd) of an f-point \(x_a\) iff there exists an f-closed (f-semi-closed) q-nbd \(U\) of \(x_a\) such that \(U \cap A'\), i.e., \(U \subseteq A\).

(2) An f-point \(x_a\) is said to be an f-semi-\(\theta\)-cluster point of an f-set \(A\) iff for every semi-open q-nbd \(U\) of \(x_a\), \(U\) is q-coincident with \(A\). The set of all f-semi-\(\theta\)-cluster points of an f-set \(A\) is called f-semi-\(\theta\)-closure of \(A\) and denoted by \([A]\). An f-set \(A\) is f-semi-\(\theta\)-closed iff \(A = [A]\). The complement of an f-semi-\(\theta\)-closed set is called an f-semi-\(\theta\)-open set.

Theorem 2.14 [15]. Let \(f: X \to Y\) be any function and \(x_a\) be any f-point in \(X\), then

(1) for \(A \subseteq X\) and \(x_a \in A\), we have \(f(x_a) \cap f(A)\),
(2) for \(B \subseteq Y\) and \(f(x_a) \subseteq B\), we have \(x_a \subseteq f^{-1}(B)\).

3. Fuzzy weakly semi-continuous functions

Definition 3.1. A mapping \(f: X \to Y\) from an fts \(X\) to an fts \(Y\) is called fuzzy weakly semi-continuous (fwsc) iff for any f-point \(x_a\) in \(X\) and any f-open set \(B\) in \(Y\) containing \(f(x_a)\), there exists an f-semi-open set \(A\) containing \(x_a\) such that \(f(A) \subseteq B\).

Note 3.2. It is clear from Theorems 2.10 and 2.7 that every f-semi-continuous function is fwsc whereas the converse is not true as can be seen from the example given below.

Example 3.3. Let \(A\), \(B\) and \(C\) be the fuzzy sets of \(I = [0, 1]\) defined as follows:

\[
A(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq 1/2, \\
2x - 1 & \text{if } 1/2 \leq x \leq 1,
\end{cases}
\]

\[
B(x) = \begin{cases} 
1 & \text{if } 0 \leq x \leq 1/4, \\
-4x + 2 & \text{if } 1/4 \leq x \leq 1/2, \\
0 & \text{if } 1/2 \leq x \leq 1,
\end{cases}
\]

\[
C(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq 1/4, \\
(4x - 1)/3 & \text{if } 1/4 \leq x \leq 1.
\end{cases}
\]

Consider the fuzzy topologies \(T_1 = \{0, C, 1\}\) and \(T_2 = \{0, 1, A, B, A \lor B\}\) on \(I\) and the mapping \(f: (I, T_1) \to (I, T_2)\) defined by \(f(x) = x/2\) for all \(x \in I\).

Then we see that \(f\) is fwsc; but \(f\) is not an f-semi-continuous function since \(f^{-1}(B) = A'\) which is not an f-semi-open set in \((I, T_1)\).

Theorem 3.4. If \(Y\) is an f-regular space, then a mapping \(f: X \to Y\) is fwsc iff \(f\) is f-semi-continuous.

Proof. The necessary part follows from Note 3.2. We prove only the sufficient part. Let \(f\) be fwsc and \(Y\) be an f-regular space. Let \(x_a\) be any f-point of \(X\) and \(B\) be any f-open set in \(Y\) containing \(f(x_a)\).

Since \(Y\) is f-regular (cf. Definition 2.12) there exists an f-open q-nbd \(C\) of \(f(x_a) = y_a\) (where \(y = f(x)\)) such that \(C \subseteq B\). Since \(f\) is fwsc and \(C\) is an f-open-q-nbd of \(f(x_a)\), there exists \(A \in \text{FSO}(X)\) with \(x_a \in A\) such that \(f(A) \subseteq C\). By Theorem 2.10, \(C \subseteq C\) and so \(f(A) \subseteq C \subseteq C \subseteq B\). Thus \(f\) is f-semi-continuous by Theorem 2.7 and this completes the proof.

In the following theorems we give some characterization of fwsc functions.

Theorem 3.5. A mapping \(f: (X, T_1) \to (Y, T_2)\) from an fts \(X\) to an fts \(Y\) is fwsc iff for each f-open set \(B\) in \(Y\), \(f^{-1}(B) \subseteq (f^{-1}(B))'\).

Proof. Let \(f\) be fwsc and \(B \in T_2\). Let \(x_a\) be an f-point in \(f^{-1}(B)\). Thus \(f(x_a) \in B\). \(f\) is fwsc implies that there exists an \(A \in \text{FSO}(X)\) such that \(x_a \in A\) and \(f(A) \subseteq B\). By Theorem 2.6(2) and (3) we have \(A \subseteq (f^{-1}(B))'\). Hence \(A' \subseteq (f^{-1}(B))'\) and since \(A\) is f-semi-open, \(A \subseteq (f^{-1}(B))'\). So \(f^{-1}(B) \subseteq A \subseteq (f^{-1}(B))'\).

Conversely let \(x_a\) be an f-point in \(X\) and \(B\) be any f-open set in \(Y\) such that \(f(x_a) \in B\). By hypothesis, \(A \subseteq (f^{-1}(B))'\). Hence \(x_a \in f^{-1}(B) \subseteq A\), which implies that \(A\) is an f-semi-open set in \(X\) containing \(x_a\). So \(A = (f^{-1}(B))' \subseteq f^{-1}(B), i.e., f(A) \subseteq B\) (cf. Theorem 2.6(4)). Hence \(f\) is fwsc and this proves the result.

Theorem 3.6. A mapping \(f: X \to Y\) from an fts \(X\) to an fts \(Y\) is fwsc if for each f-open set \(B\) in \(Y\), \(f^{-1}(B) \in \text{FSO}(X)\).

Proof. Straightforward.
The following example shows that the composition of two fwsc functions need not be fwsc.

Example 3.7. Let $I = [0, 1]$ and $E_t$ be the Euclidean subspace topology on $I$. Let $\tilde{E}_t$ be the fuzzy topology on $I$ induced by the usual topology on $I$. Let $X = Y = Z = I$ with the fuzzy topology $\tilde{E}_t$. Consider the functions $f: X \to Y$ and $g: Y \to Z$ defined by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1/2, \\ 0, & 1/2 < x \leq 1, \end{cases}$$

$$g(x) = \begin{cases} 0, & 0 \leq x < 1/2, \\ 1, & 1/2 \leq x \leq 1. \end{cases}$$

It is easy to check that $f$ and $g$ are fuzzy semi-continuous and hence fwsc (cf. Note 3.2); but $g \circ f$ is not fwsc.

Under reasonable conditions, the following result shows that $g \circ f$ is fwsc.

Theorem 3.8. If $f: X \to Y$ is $f$-irresolute and $g: Y \to Z$ is fwsc, then $g \circ f: X \to Z$ is fwsc.

Proof. Let $x_0$ be any $f$-point in $X$ and $C$ be any $f$-open set in $Z$ containing $((g \circ f)(x_0)) = g(f(x_0))$. Since $g$ is fwsc there exists an $f$-open set $B$ in $Y$ containing $f(x_0)$ such that $g(B) \subseteq \tilde{C}$. Also since $f$ is fuzzy irresolute and $B$ is $f$-open in $Y$, it follows that $f^{-1}(B)$ is $f$-semi-open in $X$. Let $A = f^{-1}(B)$. Now $(g \circ f)(A) = g(f(A)) \subseteq g(B) \subseteq \tilde{C}$. So $g \circ f$ is fwsc and this completes the proof. \qed

Corollary 3.9. If $f: X \to Y$ is $f$-semi-continuous and $f$-almost open (or $f$-open) and $g: Y \to Z$ is fwsc, then $g \circ f$ is fwsc.

Proof. The proof follows from Theorems 2.11 and 3.8. \qed

Remark 3.10. Fuzzy weakly semi-continuity of a function is both a local and global property.

4. Fuzzy weakly semi-continuous functions in terms of $q$-coincidence, $q$-neighborhoods and $\theta$-cluster points

In this section we characterize fuzzy weakly semi-continuous functions and investigate some of their properties by using the notions of quasi-coincidence, quasi-neighborhoods and $\theta$-cluster points as introduced by Ming and Ming [13].

Theorem 4.1. A mapping $f: X \to Y$ is fwsc iff corresponding to each $f$-open $q$-nbhd $B$ of $y_\gamma$ in $Y$, there exists an $f$-semi-open $q$-nbhd $A$ of $x_\gamma$ in $X$ such that $f(A) \subseteq \tilde{B}$, where $f(x_\gamma) = (f(x))_\gamma = y_\gamma$.

Proof. Let $f$ be fwsc and $B$ be an $f$-open $q$-nbhd of $y_\gamma$ where $f(x) = y$ in $Y$. So, $B(y) + \alpha > 1$. We can choose a positive real number $\gamma$ such that $B(y) + \gamma > 1 - \alpha$. Hence $B$ is an $f$-open nbhd of $y_\gamma$ in $Y$. Since $f$ is fwsc, there exists an $f$-semi-open set $A$ containing $x_\gamma$ such that $f(A) \subseteq \tilde{B}$. Now $A(x) \geq \gamma$ implies $A(x) > 1 - \alpha$, i.e., $A(x) + \alpha > 1$. Thus $x_\gamma \in A$. So $A$ is an $f$-semi-open semi-$q$-nbhd of $x_\gamma$.

Conversely, let the condition of the theorem hold, i.e., let $x_\gamma$ be an $f$-point in $X$ and $B$ be an $f$-open set in $Y$ containing $y_\gamma = (f(x))_\gamma$. So, $x_\gamma \in f^{-1}(B) = C$ (say). Hence $C(x) \geq \alpha$. We can choose a positive integer $\gamma$ such that $C(x) \geq 1/\gamma$. Put $x_\gamma = 1 + (1/n) - C(x)$, for any positive integer $n \geq \gamma$. Clearly $0 < x_\gamma < 1$ for all $n \geq \gamma$. Now $B(y) + x_\gamma = B(y) + 1 + (1/n) - C(x) = 1 + (1/n) > 1$ (since $C(x) \geq 1/(1/n)$). Hence $Y_{\gamma} \cap B$, i.e., $B$ is an $f$-open $q$-nbhd of $y_\gamma$ for all $n \geq \gamma$. So by hypothesis there exists an $f$-semi-open $q$-nbhd $A_n$ of $x_\gamma$ such that $f(A_n) \subseteq \tilde{B}$, for all $n \geq \gamma$. Now $A = \bigcup_{n \geq \gamma} A_n$ is $f$-semi-open in $X$. It remains to show that $x_\gamma \in A$. We have $A_n(x) + x_\gamma > 1$ for all $n \geq \gamma$. This implies $A_n(x) > 1 - x_\gamma = C(x) - (1/n)$ for all $n \geq \gamma$. Thus $A(x) > C(x) - (1/n)$ for all $n \geq \gamma$. Since $x_\gamma \in C$, $A(x) \geq C(x) \geq \alpha$. So $A$ is an $f$-semi-open set in $X$ such that $f(A) = f(\bigcup_{n \geq \gamma} A_n) = \bigcup_{n \geq \gamma} f(A_n) \subseteq \tilde{B}$. Hence $f$ is fwsc and this completes the proof. \qed

Lemma 4.2 [8]. For any two $f$-sets $A$ and $B$ in $X$, $A \subseteq B$ iff for each $x_\gamma$ in $X$, $x_\gamma \in A$ implies $x_\gamma \in B$. 

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Theorem 4.3. If \( f: X \rightarrow Y \) is an fwsc function from an fts \( X \) to an fts \( Y \), then for each f-open set \( B \) in \( Y \), \( f^{-1}(B) \subseteq f^{-1}(\tilde{B}) \).

Proof. An f-set is the union of all of its f-points [13]. Suppose that there is an f-point \( x_\alpha \in \tilde{f}^{-1}(B) \) but \( x_\alpha \notin f^{-1}(\tilde{B}) \). Since \( f(x_\alpha) \notin \tilde{B} \) there exists an f-open set \( C \) in \( Y \) with \( f(x_\alpha) \in C \) such that \( C \notin \tilde{B} \). Thus \( C \notin \tilde{B} \) and by Proposition 1.4 of [1] \( C \notin \tilde{B} \). Since \( f \) is fwsc, there exists an \( A \in FSO(X) \) with \( x_\alpha \in A \) such that \( f(A) \subseteq \tilde{C} \). Hence \( f(A) \notin B \), since \( f(A) \subseteq \tilde{C} \subseteq C \). But on the other hand, since each f-semi-open set is f-semi q-nbd of each of its f-point, \( x_\alpha \in \tilde{f}^{-1}(B) \) and \( A \) is an f-semi q-nbd of \( x_\alpha \). By Definition 2.9(2), \( A \subseteq f^{-1}(B) \). By Theorem 2.14(1), \( f(A) \subseteq f^{-1}(B) \), and hence by Theorem 2.6(4), \( f(A) \subseteq f^{-1}(B) \), which is a contradiction. This completes the proof of the theorem. \( \square \)

Theorem 4.4. Let \( f: X \rightarrow Y \) be an f-open and fwsc mapping. Then \( f(\tilde{A}) \subseteq f(A) \), for each f-open set \( A \) in \( X \).

Proof. Let \( A \) be an f-open set in \( X \) and let \( f(A) = B \). Since \( f \) is f-open, we see that \( B \) is an f-open set in \( Y \). Hence by Theorem 2.6(3), \( A \subseteq f^{-1}(f(A)) = f^{-1}(B) \). Since \( f \) is fwsc, we have from Theorem 4.3, \( f^{-1}(B) \subseteq f^{-1}(\tilde{B}) \). Thus \( A \subseteq f^{-1}(B) \), i.e., \( f(A) \subseteq \tilde{B} \) and this proves the result. \( \square \)

Theorem 4.5. A function \( f: X \rightarrow Y \) is fwsc iff for each f-open set \( B \) in \( Y \), \( x_\alpha \in f^{-1}(B) \) implies \( x_\alpha \in f^{-1}(\tilde{B}) \) for each f-point \( x_\alpha \) in \( X \).

Proof. Let \( f \) be fwsc. Let \( x_\alpha \) be an f-point in \( X \) and \( B \) be any f-open set in \( Y \) such that \( x_\alpha \in f^{-1}(B) \). Then \( f(x_\alpha) \in B \). Since \( f \) is fwsc by Theorem 4.1, there exists an f-semi-open set \( A \) in \( X \) such that \( x_\alpha \in A \) and \( f(A) \subseteq \tilde{B} \). By Theorem 2.6(2) and (3) we have \( A \subseteq f^{-1}(\tilde{B}) \). Since \( A \in FSO(X) \), \( A \subseteq f^{-1}(\tilde{B}) \). So \( x_\alpha \in f^{-1}(\tilde{B}) \).

Conversely, let \( f \) be fwsc. Let \( x_\alpha \) be an f-point in \( X \) and \( B \) be any f-open set in \( Y \) such that \( x_\alpha \in f^{-1}(B) \). Then \( f(x_\alpha) \in B \). Since \( f \) is fwsc, by Theorem 4.1, there exists an f-semi-open set \( A \) in \( X \) such that \( x_\alpha \in A \) and \( f(A) \subseteq \tilde{B} \). By Theorem 2.6(2) and (3) we have \( A \subseteq f^{-1}(\tilde{B}) \). Hence \( A \subseteq f^{-1}(\tilde{B}) \), i.e., \( f(A) \subseteq \tilde{B} \). Thus \( f \) is fwsc and this completes the proof. \( \square \)

Theorem 4.6. If \( f: X \rightarrow Y \) is fwsc then for each f-point \( x_\alpha \) in \( X \) and each fuzzy q-nbd \( B \) of \( f(x_\alpha) \), \( f^{-1}(B) \) is an f-semi q-nbd of \( x_\alpha \).

Proof. Let \( f: X \rightarrow Y \) be an fwsc function and \( x_\alpha \) be an f-point in \( X \). Let \( \tilde{B} \) be an f-q-nbd of \( f(x_\alpha) \). So there is an f-open q-nbd \( C \) of \( f(x_\alpha) \) such that \( \tilde{C} \subseteq \tilde{B} \). Since \( C \) is an f-open q-nbd of \( x_\alpha \), by Theorem 4.7, there is an f-semi-open q-nbd \( A \) of \( x_\alpha \) such that \( f(A) \subseteq \tilde{C} \) and thus \( f(A) \subseteq \tilde{C} \subseteq \tilde{B} \). So \( A \subseteq f^{-1}(B) \). Hence \( f^{-1}(B) \) is an f-semi q-nbd of \( x_\alpha \) and this completes the proof. \( \square \)

Theorem 4.7. If \( f: X \rightarrow Y \) is a function such that for each f-point \( x_\alpha \) in \( X \) and each f-semi q-nbd \( B \) of \( f(x_\alpha) \) in \( Y \), \( f^{-1}(B) \) is f-semi q-nbd of \( x_\alpha \), then \( f \) is fwsc.

Proof. Let \( x_\alpha \) be an f-point in \( X \) and \( B \) be any f-open q-nbd of \( f(x_\alpha) \). We note that \( B \) is an f-semi-open q-nbd of \( f(x_\alpha) \). By hypothesis, \( f^{-1}(B) \) is an f-semi q-nbd of \( x_\alpha \). So there exists an f-semi-open set \( A \) in \( X \) such that \( x_\alpha \in A \subseteq f^{-1}(B) \), i.e., \( f(A) \subseteq \tilde{B} \). Hence \( f \) is fwsc and this completes the proof. \( \square \)

Theorem 4.8. If \( f: X \rightarrow Y \) is fwsc then

(i) \( f(\tilde{A}) \subseteq [f(A)]_g \) for each f-set \( A \) in \( X \),

(ii) \( f([f^{-1}(B)]_g) \subseteq [B]_g \) for each f-set \( B \) in \( Y \).

Proof. (i) Let \( x_\alpha \in \tilde{A} \) and \( S \) be an f-closed q-nbd of \( f(x_\alpha) \). Then there exists an f-open q-nbd \( V \) of \( f(x_\alpha) \) such that \( V \subseteq S \). Since \( f \) is fwsc, by Theorem 4.1 there exists an f-semi-open q-nbd \( U \) of \( x_\alpha \) such that \( f(U) \subseteq \tilde{V} \). Since \( x_\alpha \in \tilde{A} \), by Definition 2.9(2), \( x_\alpha \) is an f-semi-cluster point of \( A \). Hence \( U \subseteq f(A) \) and also \( f(U) \subseteq \tilde{V} \), \( V \subseteq f(A) \). Again since, \( S \) is f-closed, we have \( \tilde{V} \subseteq \tilde{S} \). Therefore \( S \subseteq f(A) \). By Definition 2.13(2), \( f(x_\alpha) \in [f(A)]_g \), i.e., \( x_\alpha \in f^{-1}[f(A)]_g \). Therefore, \( A \subseteq f^{-1}[f(A)]_g \), which implies \( f(\tilde{A}) \subseteq [f(A)]_g \), proving (i).

(ii) Let \( B \) be an f-set in \( Y \) and \( x_\alpha \) be an f-point in \( X \) such that \( x_\alpha \in \tilde{f}^{-1}(B)_g \). Let \( V \) be any f-open q-nbd of \( f(x_\alpha) \). By Theorem 4.1, there exists f-semi-open q-nbd \( U \) of \( x_\alpha \) such that \( f(U) \subseteq \tilde{V} \). Since
\[
\tilde{f}^{-1}(B)^i \leq \tilde{f}^{-1}(B), \text{ we have } x_\alpha \in \tilde{f}^{-1}(B)^i. \text{ By Definition 2.13(2), } U \cap f^{-1}(B), \text{ i.e., } f(U) \cap B. \text{ Thus } \tilde{V} \cap B, \text{ which implies } f(x_\alpha) \in [B]_{\tilde{\beta}}. \text{ So } f(\tilde{f}^{-1}(B)^i) \subseteq [B]_{\tilde{\beta}}, \text{ proving (ii). } \]

5. Fuzzy weakly semi-continuous functions and fuzzy retracts

Definition 5.1. Let \( X \) be an fts and \( A \subset X \). Then the subspace (crisp) \( A \) of \( X \) is called a fuzzy retract of \( X \) if there exists a fuzzy continuous function \( r: X \rightarrow A \) such that \( r(a) = a \) for all \( a \in A \). In this case \( r \) is called a fuzzy retraction. \( A \) is called an f-open, semi-continuous retract of \( X \) if \( r \) is f-open and f-semi-continuous; similarly \( A \) is called fuzzy retract of \( X \) if \( r \) is fwsc.

Theorem 5.2. If \( A \) is an f-open, f-semi-continuous retract of the fts \( X \) then for every fts \( Y \), any fwsc function \( g: A \rightarrow Y \) can be extended to an fwsc function of \( X \) into \( Y \).

Proof. Let \( Y \) be an arbitrary fts and \( g: A \rightarrow Y \) be an fwsc function. By Corollary 3.9, \( g \circ r: X \rightarrow Y \) is fwsc and \( g \circ r(a) = g(r(a)) = g(a) \) for all \( a \in A \), where \( r: X \rightarrow A \) is an f-open, f-semi-continuous retraction. Hence \( g \circ r \) is an fwsc extension of \( g \) to \( X \) and this completes the proof.

Theorem 5.3. If \( A \) is an f-open, f-semi-continuous retract of \( X \) and \( B \) is an fwsc retract of \( A \) then \( B \) is an fwsc retract of \( X \).

Proof. Let \( r: X \rightarrow A \) be an f-open and f-semi-continuous mapping such that \( r(a) = a \) for all \( a \in A \). Let \( s: A \rightarrow B \) be an fwsc retract of \( A \) such that \( s(b) = b \) for all \( b \in B \). By Corollary 3.9, \( s \circ r: X \rightarrow B \) is fwsc and \( s \circ r(b) = b \) for all \( b \in B \). Hence \( B \) is fwsc retract of \( X \) and this proves the result.

Definition 5.4. An fts \( X \) is called an f-quasi Urysohn space if for any two distinct fuzzy points \( x_\alpha \) and \( y_\beta \), there exist f-open sets \( U_1 \) and \( U_2 \) in \( X \) such that \( x_\alpha \in U_1, \ y_\beta \in U_2 \) and \( U_1 \cap U_2 = \emptyset \).

Definition 5.5. An fts \( X \) is said to be fuzzy quasi semi-Hausdorff if distinct fuzzy points in \( X \) have disjoint semi q-nbds, i.e., if \( x_\alpha \) and \( y_\beta \) are distinct fuzzy points in \( X \), then there exist fuzzy semi q-nbds \( V_1 \) and \( V_2 \) such that \( x_\alpha \in V_1, \ y_\beta \in V_2 \) and \( V_1 \cap V_2 = \emptyset \).

Theorem 5.6. If \( Y \) is an f-quasi Urysohn space and \( f: X \rightarrow Y \) is an fwsc injection, then \( X \) is a fuzzy quasi semi-Hausdorff space.

Proof. Let \( x_\alpha \) and \( y_\beta \) be two distinct fuzzy points in \( X \). \( f \) being injective, \( f(x_\alpha) \) and \( f(y_\beta) \) are distinct fuzzy points in \( Y \). Since \( Y \) is fuzzy quasi Urysohn, there exists f-open sets \( V_1 \) and \( V_2 \) in \( Y \) such that \( f(x_\alpha) \in V_1, \ f(y_\beta) \in V_2 \) and \( V_1 \cap V_2 = \emptyset \), i.e., \( f^{-1}(V_1)^i \land f^{-1}(V_2)^i = \emptyset \). By Theorem 3.5, \( x_\alpha \in f^{-1}(V_1)^i \subseteq f^{-1}(\tilde{V}_1)^i \subseteq f^{-1}(\tilde{V}_1)^i \).

Similarly \( y_\beta \in f^{-1}(V_2)^i \subseteq f^{-1}(\tilde{V}_2)^i \subseteq f^{-1}(\tilde{V}_2)^i \).

So, \( f^{-1}(\tilde{V}_1)^i \) and \( f^{-1}(\tilde{V}_2)^i \) are disjoint fuzzy semi q-nbds of \( x_\alpha \) and \( y_\beta \), respectively. So \( X \) is fuzzy quasi semi-Hausdorff and this proves the result.

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References

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