INTERVAL WAVELET SETS DETERMINED BY POINTS ON THE CIRCLE

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Abstract: Having observed that an interval wavelet set corresponds to the points in a circle, we obtain points in the circle which characterize two-interval wavelet sets and also those points which characterize three-interval wavelet sets for dilation $d \geq 2$. Further points in the circle characterizing one-interval and two-interval H^2-wavelet sets for dilation $d \geq 2$ are obtained. In addition, we discuss three-interval wavelet sets of $\mathbb{R}$ in respect of being associated with a multiresolution analysis.

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1. Introduction

Dai and Larson [5] called a measurable set W of the real line $\mathbb{R}$ to be a *wavelet set* if the characteristic function χ_W on W is equal to the modulus of the Fourier transform $\hat{\psi}$ of some orthonormal wavelet ψ on $L^2(\mathbb{R})$. By an *orthonormal wavelet* ψ, we mean a function in $L^2(\mathbb{R})$, whose successive dilates by a scalar d other than 0, 1 and -1, followed by all integral translates, form an orthonormal basis for $L^2(\mathbb{R})$. These definitions were generalized to higher dimensions in [2, 4, 6, 7, 13]. Fang and Wang in [8] introduced a *minimally supported frequency* (MSF) *wavelet*, the Fourier transform of which has support of smallest possible measure. MSF wavelets are indeed those wavelets which are associated with wavelet sets.

One of the earliest wavelets namely Shannon or Littlewood-Paley wavelet for dilation 2 in $L^2(\mathbb{R})$ has $W = [-2\pi\,,\,-\pi] \cup [\pi\,,\,2\pi]$ as its wavelet set. It is the union of two disjoint intervals of $\mathbb{R}$. Ha, Kang, Lee and Seo [9] characterized wavelet sets for dilation 2 in $\mathbb{R}$ which are unions of two disjoint intervals and also those which are unions of three disjoint intervals. Those with two intervals are precisely

$$[2a - 4\pi,\, a - 2\pi] \cup [a,\, 2a],$$

for some $0 < a < 2\pi$, while those with three intervals are

$$W(j,p) \equiv \left[-2\left(1 - \frac{2p+1}{2^{j+1}-1}\right)\pi\,,\, -\left(1 - \frac{2p+1}{2^{j+1}-1}\right)\pi\right] \cup \left[\frac{2(p+1)\pi}{2^{j+1}-1}\,,\,\frac{2(2p+1)\pi}{2^{j+1}-1}\right]$$
$$\cup \left[\frac{2^{j+1}(2p+1)\pi}{2^{j+1}-1}\,,\,\frac{2^{j+2}(p+1)\pi}{2^{j+1}-1}\right]$$

for natural numbers j and p such that $j \geq 2$ and $1 \leq p \leq 2^j - 2$, together with $-W(j,p)$. Further, it has been shown that each of the two-interval wavelet sets is associated with a multiresolution analysis (MRA) while in case of three-interval wavelet sets it is found that for odd p and any j it is not associated with an MRA. It is pertinent to add that Ionascu [11] introduced the notion of wavelet induced isomorphism to obtain a characterization of wavelet sets with the help of which we reformulated the characterization of two-interval as well as three-interval wavelet sets in [14].

Also, Bownik and Hoover [3] characterized two-interval and three-interval wavelet sets for dilation d greater than 1.

Determining wavelet sets of $\mathbb{R}$ which are unions of intervals remained a matter of interest which got investigated in various papers [1, 3, 5, 9].

The collection $H^2(\mathbb{R})$ of all square integrable functions whose Fourier transforms are supported in $(0, \infty)$ is called the *Hardy space* and an element $\psi \in H^2(\mathbb{R})$ for which the family $\{\psi_{j,k} \equiv 2^{j/2}\psi(2^j \cdot -k) : j, k \in \mathbb{Z}\}$ forms an orthonormal basis for $H^2(\mathbb{R})$, is said to be an H^2-*wavelet*. Similar to the L^2-case, an H^2-wavelet ψ will be called an MSF *wavelet* if $|\hat{\psi}| = \chi_K$ for some measurable set $K \subseteq (0, \infty)$. The associated K is called an H^2-*wavelet set*. In this case, as well, the Lebesgue measure $\mu(K)$ of K is 2π.

Since a wavelet set is 2π-translation congruent to an interval of Lebesgue measure 2π, a wavelet set having finitely many components can on different translations of different components partition $[0, 2\pi)$ a.e.. Thus we have points in the circle, in number equal to the number of components in the wavelet set, which determine the wavelet set. Similar is the situation when the wavelet set has infinitely many components. However, the converse need not be true. For example any three points in the circle need not necessarily provide a three-interval wavelet set. Also a pair of points in the circle, equivalently in $(0, 2\pi]$, containing 2π does not provide a two-interval wavelet set.

In Section 3 of this article, we determine the class of those two points in the circle which provide wavelet sets of $\mathbb{R}$ for dilation 2 with two intervals. A class of three points in the circle has been determined providing three-interval wavelet sets for dilation $d \geq 2$, in Section 4. The technique involves sets with Lebesgue measure 2π determined by an element of the unit circle, the parts of which when translated suitably by integral multiples of 2π on the two sides of the real line yield desired wavelet sets. Because the sets after such integral translations remain 2π-translation congruent to $[0, 2\pi)$, only dilations determine these wavelet sets. Indeed, the process re-characterizes such wavelet sets. Similar results are obtained for one-interval and two-interval H^2-wavelet sets, in Section 5, which re-characterize such H^2-wavelet sets for dilation 2 obtained in [9]. Further, using a result in [8], we provide an alternative proof of the fact that a three-interval wavelet set for odd p and any $j \geq 2$ is not associated with an MRA in the last Section 6. In addition, we prove that if $p = 2^j - 2$, then the wavelet set $W(j, p)$ is associated with an MRA, where $j \geq 2$.

2. Preliminaries

Two measurable sets E and F of $\mathbb{R}$ are said to be *translation congruent modulo* 2π if there is a measurable bijection $\tau : E \longrightarrow F$ such that $\tau(s) - s$ is an integral multiple of 2π for each $s \in E$; or equivalently, if there is a measurable partition $\{E_n : n \in \mathbb{Z}\}$ of E such that $\{E_n + 2n\pi : n \in \mathbb{Z}\}$ is a measurable partition of F. We call τ to be a 2π-*translation map*. Analogously, measurable sets E and F are *dilation congruent modulo* 2 if there is a measurable bijection $\delta : E \longrightarrow F$ such that for each $s \in E$, there is an integer n such that $\delta(s) = 2^n s$; or equivalently, if there is a measurable partition $\{E_n : n \in \mathbb{Z}\}$ of E such that $\{2^n E_n : n \in \mathbb{Z}\}$ is a measurable partition of F.

A measurable set $E \subseteq \mathbb{R}$ is translation congruent to $[0, 2\pi)$ modulo 2π iff the sets $E + 2n\pi \equiv \{s + 2n\pi : s \in E\}$, $n \in \mathbb{Z}$, are pairwise disjoint and $\mathbb{R} \setminus \bigcup_{n\in\mathbb{Z}} (E + 2n\pi)$ is a null set. Also, a measurable set $E \subseteq \mathbb{R}$ is dilation congruent modulo 2 to the set $[-2\pi, -\pi) \cup [\pi, 2\pi)$ iff the sets $2^n E \equiv \{2^n s : s \in E\}$, $n \in \mathbb{Z}$ are pairwise disjoint and $\mathbb{R} \setminus \bigcup_{n\in\mathbb{Z}} 2^n E$ is a null set. For other related notion, see [5, 10].

Among several criteria available for a measurable set $E \subseteq \mathbb{R}$ to be a wavelet set, [2, 5, 9, 11, 12] the one we shall use is quoted below:

Let $E \subseteq \mathbb{R}$ be a measurable set. Then E is a wavelet set iff E is both 2π-translation congruent to $[0, 2\pi)$ modulo 2π and dilation congruent modulo 2 to $[-2\pi, -\pi) \cup [\pi, 2\pi)$; or equivalently, E

is a wavelet set iff

(a) $\mathbb{R} = \dot{\bigcup}_{n\in\mathbb{Z}} (E + 2n\pi)$, a.e.,

(b) $\mathbb{R} = \dot{\bigcup}_{n\in\mathbb{Z}} 2^n E$, a.e.,

where $\dot{\bigcup}$ denotes the disjoint union.

For a set W in $\mathbb{R}$, W^+ denotes $W \cap [0\,,\,\infty)$ and W^- denotes $W \cap (-\infty,\,0]$. Since W is a wavelet set of $\mathbb{R}$ iff $-W \equiv \{-w : w \in W\}$ is a wavelet set of $\mathbb{R}$, we shall consider three-interval wavelet sets of $\mathbb{R}$ for which W^+ has two components. In the sequel, we denote $(0\,,\,\infty)$ by $\mathbb{R}^+$ and $(-\infty\,,\,0)$ by $\mathbb{R}^-$.

A set $K \subseteq \mathbb{R}^+$ is an H^2-wavelet set iff the following two conditions hold:

(a) $\mathbb{R} = \dot{\bigcup}_{n\in\mathbb{Z}} (K + 2n\pi)$, a.e.,

(b) $\mathbb{R}^+ = \dot{\bigcup}_{n\in\mathbb{Z}} 2^n K$, a.e..

Consider the map p from $\mathbb{R}$ to S^1 which sends $t \in \mathbb{R}$ to $e^{it} \in S^1$. We shall identify t in $(0,\,2\pi]$ with e^{it}. For $\alpha, \beta, \gamma \in S^1$,

(i) $p^{\leftarrow}(\alpha)$ denotes $[\alpha\,,\,\alpha + 2\pi]$, where $0 < \alpha \leq 2\pi$.

(ii) $p^{\leftarrow}(\alpha\,,\,\beta)$ denotes $[\alpha\,,\,\beta] \cup [\beta\,,\,\alpha + 2\pi]$, where $0 < \alpha < \beta \leq 2\pi$.

(iii) $p^{\leftarrow}(\alpha\,,\,\beta\,,\,\gamma)$ denotes $[\alpha\,,\,\beta] \cup [\beta\,,\,\gamma] \cup [\gamma\,,\,\alpha + 2\pi]$, where $0 < \alpha < \beta < \gamma \leq 2\pi$.

3. Two-Interval Wavelet Sets

Let $\alpha, \beta \in S^1$ be such that $0 < \alpha < \beta \leq 2\pi$. Then

$$p^{\leftarrow}(\alpha, \beta) = [\alpha\,,\,\beta] \cup [\beta\,,\,\alpha + 2\pi]\,.$$

We obtain two-interval wavelet sets of $\mathbb{R}$ by translating intervals $[\alpha\,,\,\beta]$ and $[\beta\,,\,\alpha + 2\pi]$ on opposite sides by integral multiples of 2π. Consider the translates $[\alpha\,,\,\beta] + 2l\pi$ and $[\beta\,,\,\alpha + 2\pi] - 2k\pi$, where $l \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N} - \{1\}$, to obtain a wavelet set denoted by $W_{(\alpha\,,\,\beta)}$ or by W, if there is no confusion. In fact, W^- would then be $[\beta - 2k\pi\,,\,\alpha + 2\,(1-k)\,\pi]$ while W^+ be $[\alpha + 2l\pi\,,\,\beta + 2l\pi]$. Noting that $W \equiv W^- \cup W^+$ is 2π-translation congruent to $[0, 2\pi)$, we obtain the values of l and k from the conditions that

$$\text{(i) } \dot{\bigcup}_{j\in\mathbb{Z}} 2^j W^+ = \mathbb{R}^+, \quad \text{and (ii) } \dot{\bigcup}_{j\in\mathbb{Z}} 2^j W^- = \mathbb{R}^-,$$

ensuring W to be a wavelet set.

Since the interval $[\beta + 2l\pi\,,\,2\,(\alpha + 2l\pi)]$ remains uncovered by the family $\{2^j W^+ : j \in \mathbb{Z}\}$ if $\beta + 2l\pi < 2\,(\alpha + 2l\pi)$, and that $W^+ \cap 2W^+$ has non-zero measure if $\beta + 2l\pi > 2\,(\alpha + 2l\pi)$, from (i) we obtain

$$2\alpha - \beta = -2l\pi. \tag{1}$$

A similar argument applied to (ii) gives

$$2\alpha - \beta = 2k\pi - 4\pi. \tag{2}$$

From (1) and (2), we obtain $k + l = 2$ and hence $l = 0$ and $k = 2$, in view of the fact that $k \geq 2$. Thus $\beta = 2\alpha$, and the wavelet set W is

$$[2\alpha - 4\pi\,,\,\alpha - 2\pi] \cup [\alpha\,,\,2\alpha],$$

where $\alpha \leq \pi$. Notice that the Lebesgue measure $\mu(W^+)$ of W^+ is $\alpha \leq \pi$ while $\mu(W^-) \geq \pi$. These points α, β of S^1 determine another wavelet set $-W$. It may be noted that the pair of points α, β in S^1 gives rise to exactly two two-interval wavelet sets, except when $\alpha = \pi$, in which case $W = -W$, providing the Shannon wavelet set.

Remark 3.1. Following similar lines as above, we find that there exist no wavelet set for dilation $d > 2$ consisting of two intervals as no k and l are obtainable.

Remark 3.2. It may be noticed that only dilations do the job as we begin with sets which are 2π-translation congruent to $[0, 2\pi)$. These sets get translated on the two sides of the real line and hence the resulting sets remain 2π-translation congruent to $[0, 2\pi)$.

4. Three-Interval Wavelet Sets

Choose three elements α, β, γ in S^1 such that $0 < \alpha < \beta < \gamma \leq 2\pi$. In this Section, we determine α, β, γ which produce three-interval wavelet sets. Recall that

$$p^{\leftarrow}(\alpha, \beta, \gamma) = [\alpha\,,\, \beta] \cup [\beta\,,\, \gamma] \cup [\gamma\,,\, \alpha + 2\pi]\,.$$

In view of the fact that W is a wavelet set iff $-W$ is a wavelet set, we obtain those wavelet sets for which W^- consists of only one interval.

For a natural number j we introduce the following notation,

$$F_j = \left\{0,\ 1,\ 2,\ \ldots,\ \left[d^j - 1\right]\right\},$$

where d is a real number greater than 2 and $[r]$ denotes the integral part of a real number r.

The following Theorem determines all three-interval wavelet sets of $\mathbb{R}$ for dilation $d > 2$.

Theorem 4.1. *Let $j \in \mathbb{N}$ and $m \in F_j$ such that $m < d^j - 1$. Then the points*

$$\alpha = \frac{2(m+1)\pi}{d^{j+1} - 1},\ \beta = \frac{(md + d - 1)}{(m+1)}\alpha \ \text{ and } \ \gamma = \frac{\left(m + d^j(d-1)\right)}{(m+1)}\alpha$$

in S^1 are such that

(1) $0 < \alpha < \beta < \gamma < 2\pi$, and

(2) $p^{\leftarrow}(\alpha\,,\, \beta\,,\, \gamma) = [\alpha\,,\, \beta] \cup [\beta\,,\, \gamma] \cup [\gamma\,,\, \alpha + 2\pi]$ *determines a three-interval wavelet set for dilation d by the translation of $[\beta\,,\, \gamma]$ on the left by -2π and that of $[\gamma\,,\, \alpha + 2\pi]$ on the right by $2m\pi$.*

Proof. Let α, β, γ be points in S^1 such that $\alpha < \beta < \gamma$. Without any loss of generality we may assume that $\alpha > 0$. Consider, $p^{\leftarrow}(\alpha\,,\, \beta\,,\, \gamma) = [\alpha\,,\, \beta] \cup [\beta\,,\, \gamma] \cup [\gamma\,,\, \alpha + 2\pi]$. We have the following three cases:

Case I.

$$W^- = [\alpha - 2k\pi\,,\, \beta - 2k\pi]\,;$$

$$W^+ = [\beta + 2l\pi\,,\, \gamma + 2l\pi] \cup [\gamma + 2m\pi\,,\, \alpha + 2\pi + 2m\pi]\,,$$

where $k, l, m \in \mathbb{N} \cup \{0\}$ and $k \geq 1$.

Case II.

$$W^- = [\beta - 2k\pi\,,\, \gamma - 2k\pi]\,;$$

$$W^+ = [\alpha + 2l\pi\,,\, \beta + 2l\pi] \cup [\gamma + 2m\pi\,,\, \alpha + 2\pi + 2m\pi]\,,$$

where $k, l, m \in \mathbb{N} \cup \{0\}$ and $k \geq 1$, if $\gamma < 2\pi$ and in case $\gamma = 2\pi$, $k \geq 2$.

Case III.

$$W^- = [\gamma - 2k\pi\,,\, \alpha + 2\pi - 2k\pi]\,;$$
$$W^+ = [\alpha + 2l\pi\,,\, \beta + 2l\pi] \cup [\beta + 2m\pi\,,\, \gamma + 2m\pi]\,,$$

where $k, l, m \in \mathbb{N} \cup \{0\}$ and $k \geq 2$.

Evidently, in each of the above cases $W \equiv W^- \cup W^+$ is 2π-translation congruent to $[0,\, 2\pi)$. Suppose that $W^+ = [a\,,\, b] \cup [c\,,\, e]$, where $0 < a < b < c < e$. In order that $\dot{\bigcup}_{j\in\mathbb{Z}} d^j W^+ = \mathbb{R}^+$, we should have $[b\,,\, c] = \dot{\bigcup}_{j\in A} d^j W^+$, for some $A \subset \mathbb{Z} \setminus \{0\}$. Therefore, either $d^j b = c$ or $d^{-j} e = c$, for some $j \in \mathbb{N}$. If $d^{-j} e = c$, then j has to be equal to 1 and in that case, $\mathbb{R}^+ = \dot{\bigcup}_{j\in\mathbb{Z}} d^j\,[c\,,\, e] = \dot{\bigcup}_{j\in\mathbb{Z}} d^j\,[e/d\,,\, e]$. Thus $\mathbb{R}^+$ gets disjointly covered only by the dilates of $[c\,,\, e]$, which is not desired. Therefore, $d^j b = c$ for some $j \in \mathbb{N}$. Similarly, $d^k a = e$ for some $k \in \mathbb{N}$. In fact, if $d^j b = c$, then $k = j + 1$. Hence, $d^{j+1} a = e$. Thus

$$W^+ = [a\,,\, b] \cup \left[d^j b\,,\, d^{j+1} a\right]\,, \quad \text{where} \;\; j \in \mathbb{N}. \tag{3}$$

Observe that

$$\left[a\,,\, d^{j+1} a\right] = \dot{\bigcup}_{k=0}^{j} \left(d^{j-k}\,[a\,,\, b] \cup d^{-k}\,[c\,,\, e]\right)$$

and $\mathbb{R}^+ = \dot{\bigcup}_{m\in\mathbb{Z}} d^{mk}\left[\alpha\,,\, d^k\alpha\right]$ for $k \geq 1$ and $\alpha > 0$.
Therefore, W^+ given by (4.1) satisfies $\dot{\bigcup}_{j\in\mathbb{Z}} d^j W^+ = \mathbb{R}^+$. That

$$W^- = [dc\,,\, c]\,, \quad \text{for some} \;\; c < 0, \tag{4}$$

follows as the dilates of W^- have to disjointly cover $\mathbb{R}^-$.

Case I. When

$$W^- = [\alpha - 2k\pi\,,\, \beta - 2k\pi]$$

and

$$W^+ = [\beta + 2l\pi\,,\, \gamma + 2l\pi] \cup [\gamma + 2m\pi\,,\, \alpha + 2\pi + 2m\pi],$$

where $k, l, m \in \mathbb{N} \cup \{0\}$ and $k \geq 1$, the intervals in W^+ are disjoint iff $m \neq l$.

Suppose $l < m$. Then

$$W = [\alpha - 2k\pi\,,\, \beta - 2k\pi] \cup [\beta + 2l\pi\,,\, \gamma + 2l\pi] \cup [\gamma + 2m\pi\,,\, \alpha + 2\,(m+1)\,\pi].$$

From (3) and (4), we obtain

(a) $\alpha - 2k\pi = d\,(\beta - 2k\pi)$,
(b) $d^j\,(\gamma + 2l\pi) = \gamma + 2m\pi$, and
(c) $d^{j+1}\,(\beta + 2l\pi) = \alpha + 2\,(m+1)\,\pi$.

Hence

$$\alpha = \frac{2\left(m + 1 - d^{j+1} l - d^j\,(d-1)\,k\right)\pi}{d^j - 1},$$
$$\beta = \frac{2\left(m + k + 1 - d^{j+1} l - dk\right)\pi}{d\,(d^j - 1)},$$

and

$$\gamma = \frac{2\left(m - d^j l\right)\pi}{d^j - 1}.$$

Since $\alpha > 0$, $m + 1 > d^j\,(ld + k\,(d-1))$. Further, since $\gamma \leq 2\pi$, $m + 1 \leq d^j\,(l+1)$. Hence, $l + k < 1/(d-1)$, which is not possible. Next, suppose that $l > m$, then

$$W = [\alpha - 2k\pi\,,\,\beta - 2k\pi] \cup [\gamma + 2m\pi\,,\,\alpha + 2\,(m+1)\,\pi] \cup [\beta + 2l\pi\,,\,\gamma + 2l\pi],$$

where $k \geq 1$ and $l, m \in \mathbb{N} \cup \{0\}$.
From (3) and (4), we get

$$\beta = \frac{2\left(l - d^j k - d^j m - d^j + d^{j+1} k\right)\pi}{d^{j+1} - 1}$$

and

$$\gamma = \frac{2\left(l - d^{j+1} m\right)\pi}{d^{j+1} - 1}$$

As $\beta < \gamma$, $m + k < 1/(d-1)$, which is not possible.
Case II. When

$$W^- = [\beta - 2k\pi\,,\,\gamma - 2k\pi]$$

and

$$W^+ = [\alpha + 2l\pi\,,\,\beta + 2l\pi] \cup [\gamma + 2m\pi\,,\,\alpha + 2\pi + 2m\pi],$$

where $k \geq 1$ and $l, m \in \mathbb{N} \cup \{0\}$, if $\gamma < 2\pi$ and in case $\gamma = 2\pi$, $k \geq 2$, for the intervals in W^+ to be disjoint we should have either $l < m + 1$, or $l > m + 1$.
Suppose that $l < m + 1$. Then

$$W = [\beta - 2k\pi\,,\,\gamma - 2k\pi] \cup [\alpha + 2l\pi\,,\,\beta + 2l\pi] \cup [\gamma + 2m\pi\,,\,\alpha + 2\,(m+1)\,\pi].$$

From (3) and (4), we find that

$$\alpha = \frac{2\left(m + 1 - d^{j+1} l\right)\pi}{d^{j+1} - 1},$$

$$\beta = \frac{2\left(dm - d^{j+1} l + dk - k\right)\pi}{d^{j+1} - 1},$$

and

$$\gamma = \frac{2\left(m - d^j l + d^{j+1} k - k d^j\right)\pi}{d^{j+1} - 1}.$$

Since $\alpha > 0$, $m + 1 > d^{j+1} l$. Further, since $\gamma \leq 2\pi$, $m + 1 \leq d^j\,(d + l - k\,(d-1))$. Therefore, $l + k < d/(d-1)$, which gives $l = 0$, $k = 1$ and hence the case $\gamma = 2\pi$ which requires $k \geq 2$ is not possible. Thus

$$\alpha = \frac{2\,(m+1)\,\pi}{d^{j+1} - 1};\ \beta = \frac{2\,(md + d - 1)\,\pi}{d^{j+1} - 1};\ \gamma = \frac{2\left(m + d^{j+1} - d^j\right)\pi}{d^{j+1} - 1}$$

and

$$W = [\beta - 2\pi\,,\,\gamma - 2\pi] \cup [\alpha\,,\,\beta] \cup [\gamma + 2m\pi\,,\,\alpha + 2\pi + 2m\pi]\,.$$

Since $\beta < \gamma$, $m < d^j - 1$. Thus $j \geq 1$ and $0 \leq m < d^j - 1$. Hence, the resulting wavelet set is

$$W = \left[\frac{2\left(md + d - d^{j+1}\right)\pi}{d^{j+1} - 1}\,,\,\frac{2\left(m + 1 - d^j\right)\pi}{d^{j+1} - 1}\right] \cup \left[\frac{2(m+1)\pi}{d^{j+1} - 1}\,,\,\frac{2(md + d - 1)\pi}{d^{j+1} - 1}\right]$$
$$\cup \left[\frac{2d^j(md + d - 1)\pi}{d^{j+1} - 1}\,,\,\frac{2d^{j+1}(m+1)\pi}{d^{j+1} - 1}\right]$$

with $j \geq 1$ and $0 \leq m < d^j - 1$.
Suppose, next that $l > m + 1$. Then

$$W = [\beta - 2k\pi\,,\,\gamma - 2k\pi] \cup [\gamma + 2m\pi\,,\,\alpha + 2\,(m+1)\,\pi] \cup [\alpha + 2l\pi\,,\,\beta + 2l\pi].$$

Now, (3) and (4) give

$$\alpha = \frac{2\left(l - d^j - md^j\right)\pi}{d^j - 1},$$

and

$$\beta = \frac{2\left(l - d^{j+1}m + d^j k - d^{j+1}k\right)\pi}{d^j - 1}.$$

As $\alpha < \beta$, $m + k < 1/\left(d - 1\right)$, which is not possible. Therefore, in this case, W is not a wavelet set.

Case III. When

$$W^- = [\gamma - 2k\pi\,,\, \alpha + 2\pi - 2k\pi]$$

and

$$W^+ = [\alpha + 2l\pi\,,\, \beta + 2l\pi] \cup [\beta + 2m\pi\,,\, \gamma + 2m\pi],$$

intervals in W^+ will be disjoint iff either $m < l$, or $m > l$. Similar to the cases considered above we can find α, β and γ with the help of (3) and (4). But, on applying the condition $0 < \alpha < \beta < \gamma \le 2\pi$, we find that neither $m < l$, nor $m > l$ provides a wavelet set. □

With slight modification in the proof of the above Theorem, setting $E_j = \left\{1,\ 2,\ \ldots,\ 2^j - 2\right\}$ for a $j \in \mathbb{N} - \{1\}$, we obtain the following characterization of three-interval wavelet sets for dilation 2.

Theorem 4.2. *Let $(j, m) \in \bigcup_{k \ge 2}\{k\} \times E_k$. Then the points*

$$\alpha = \frac{2\left(m+1\right)\pi}{2^{j+1} - 1},\ \beta = \frac{(2m+1)}{(m+1)}\alpha \ \textit{and}\ \gamma = \frac{\left(m + 2^j\right)}{(m+1)}\alpha$$

in S^1 are such that

(1) $0 < \alpha < \beta < \gamma < 2\pi$, *with* $\alpha < \pi$ *and* $\gamma > \pi$, *and*

(2) $p^{\leftarrow}\left(\alpha\,,\,\beta\,,\,\gamma\right) = [\alpha\,,\,\beta] \cup [\beta\,,\,\gamma] \cup [\gamma\,,\,\alpha + 2\pi]$ *determines a three-interval wavelet set by the translation of $[\beta\,,\,\gamma]$ on the left by -2π and that of $[\gamma\,,\,\alpha + 2\pi]$ on the right by $2m\pi$.*

Remark 4.1. Above Theorems re-characterize three-interval wavelet sets obtained for $d \ge 2$ in [3].

Remark 4.2. Remark 3.2 remains in force here as well.

5. One-Interval and Two-Interval H^2-Wavelet Sets

Choose an $\alpha \in S^1$ such that $0 < \alpha \le 2\pi$. Then $p^{\leftarrow}\left(\alpha\right) = [\alpha\,,\,\alpha + 2\pi]$. That $W = [\alpha\,,\,\alpha + 2\pi]$ satisfies $\mathbb{R} = \dot{\bigcup}_{n\in\mathbb{Z}}\left(W + 2n\pi\right)$ is obvious. To have $\mathbb{R}^+ = \dot{\bigcup}_{n\in\mathbb{Z}} 2^n W$, we require $2\alpha = \alpha + 2\pi$ which provides $\alpha = 2\pi$. Therefore, $[2\pi\,,\,4\pi]$ is the only one-interval H^2-wavelet set.

For dilation $d > 2$, the only one-interval H^2-wavelet set can be found to be

$$\left[\frac{2\pi}{d-1}\,,\,\frac{2d\pi}{d-1}\right].$$

For $j \in \mathbb{N}$, $d > 2$, set $F'_{j_1} = \{1,\ 2,\ \ldots,\ [d^j - 1]\}$, and

$$F'_{j_2} = \left\{ [d^j]\,,\ [d^j] + 1,\ \ldots,\ \left[\frac{d\left(d^j - 1\right)}{d-1}\right]\right\}.$$

The following Theorem determines all two-interval H^2-wavelet sets of $\mathbb{R}$ for dilation $d > 2$.

Theorem 5.1. (1) *For* $m \in F'_{j_1}$, *the points*

$$\alpha = \frac{2\,(m+1)\,\pi}{d^{j+1}-1} \quad \textit{and} \quad \beta = \frac{2m\pi}{d^j - 1}$$

in S^1 *are such that* $0 < \alpha < \beta \le 2\pi$, *and* $[\alpha\,,\,\beta] \cup [\beta + 2m\pi\,,\,\alpha + 2\pi + 2m\pi]$ *forms an* H^2*-wavelet set.*
(2) *For* $m \in F'_{j_2}$ *such that* $m < \frac{d(d^j-1)}{d-1}$, *the points*

$$\alpha = \frac{2\left(m+1-d^j\right)\pi}{d^j - 1} \quad \textit{and} \quad \beta = \frac{2\,(m+1)\,\pi}{d^{j+1}-1}$$

in S^1 *are such that* $0 < \alpha < \beta \le 2\pi$, *and* $[\beta\,,\,\alpha + 2\pi] \cup [\alpha + 2\,(m+1)\,\pi\,,\,\beta + 2\,(m+1)\,\pi]$ *forms an* H^2*-wavelet set.*
Proof. First consider the translate of $[\alpha\,,\,\beta]$ by $2l\pi$ and that of $[\beta\,,\,\alpha+2\pi]$ by $2m\pi$, where $l, m \in \mathbb{N} \cup \{0\}$ and set

$$W = [\alpha + 2l\pi\,,\,\beta + 2l\pi] \cup [\beta + 2m\pi\,,\,\alpha + 2\pi + 2m\pi]\,.$$

The two intervals in W will be disjoint iff $l < m$, or $l > m+1$ and in that case W satisfies $\mathbb{R} = \dot{\bigcup}_{n\in\mathbb{Z}} (W + 2n\pi)$. If $l < m$, then

$$W = [\alpha + 2l\pi\,,\,\beta + 2l\pi] \cup [\beta + 2m\pi\,,\,\alpha + 2\pi + 2m\pi]\,,$$

and from (3), we have

$$d^{j+1}\,(\alpha + 2l\pi) = \alpha + 2\,(m+1)\,\pi, \quad \text{and} \quad d^j\,(\beta + 2l\pi) = \beta + 2m\pi,$$

for some $j \in \mathbb{N}$.
Therefore,

$$\alpha = \frac{2\left(m+1-d^{j+1}l\right)\pi}{d^{j+1}-1} \quad \text{and} \quad \beta = \frac{2\left(m - d^j l\right)\pi}{d^j - 1}.$$

Since $\alpha > 0$, $m+1 > d^{j+1}l$ and as $\beta \le 2\pi$, $m+1 \le d^j\,(l+1)$. Therefore, we have $l < 1/(d-1)$. Hence, $l = 0$.
Now, since $\beta \le 2\pi$, $m \le \left(d^j - 1\right)$, and $l < m$ gives that $0 < m$. Therefore,

$$\alpha = \frac{2\,(m+1)\,\pi}{d^{j+1}-1} \quad \text{and} \quad \beta = \frac{2m\pi}{d^j - 1}.$$

Thus

$$W = \left[\frac{2\,(m+1)\,\pi}{d^{j+1}-1}\,,\,\frac{2m\pi}{d^j-1}\right] \cup \left[\frac{2d^j m\pi}{d^j-1}\,,\,\frac{2d^{j+1}\,(m+1)\,\pi}{d^{j+1}-1}\right],$$

where $j \in \mathbb{N}$ and $0 < m \le \left(d^j - 1\right)$.
In case $l > m+1$, we have

$$W = [\beta + 2m\pi\,,\,\alpha + 2\pi + 2m\pi] \cup [\alpha + 2l\pi\,,\,\beta + 2l\pi]\,.$$

As before, (3) gives

$$\alpha = \frac{2\left(l - d^j - d^j m\right)\pi}{d^j - 1} \quad \text{and} \quad \beta = \frac{2\left(l - d^{j+1} m\right)\pi}{d^{j+1} - 1}.$$

On applying the condition $0 < \alpha < \beta \leq 2\pi$, we get $m = 0$ and $\left(d^j - 1\right) < l - 1 < \frac{d\left(d^j - 1\right)}{d-1}$.

Thus

$$W = \left[\frac{2\,(p+1)\,\pi}{d^{j+1} - 1}\,,\ \frac{2p\pi}{d^j - 1}\right] \cup \left[\frac{2d^j p\pi}{d^j - 1}\,,\ \frac{2d^{j+1}\,(p+1)\,\pi}{d^{j+1} - 1}\right],$$

where $p = l - 1$, $j \in \mathbb{N}$ and $\left(d^j - 1\right) < p < \frac{d\left(d^j - 1\right)}{d-1}$. □

A similar procedure, gives the following result for two-interval H^2-wavelet sets, in case of dilation $d = 2$.

For $j \in \mathbb{N}$, define $E'_{j_1} = \left\{1,\ 2,\ \ldots,\ 2^j - 1\right\}$, and $E'_{j_2} = \left\{2^j,\ 2^j + 1,\ \ldots,\ 2^{j+1} - 3\right\}$.

Theorem 5.2. (1) *For $m \in E'_{j_1}$, the points*

$$\alpha = \frac{2\,(m+1)\,\pi}{2^{j+1} - 1} \quad \textit{and} \quad \beta = \frac{2m\pi}{2^j - 1}$$

in S^1 are such that $0 < \alpha < \beta \leq 2\pi$, and $[\alpha\,,\,\beta] \cup [\beta + 2m\pi\,,\,\alpha + 2\pi + 2m\pi]$ forms an H^2-wavelet set.

(2) *For $m \in E'_{j_2}$, the points*

$$\alpha = \frac{2\left(m + 1 - 2^j\right)\pi}{2^j - 1} \quad \textit{and} \quad \beta = \frac{2\,(m+1)\,\pi}{2^{j+1} - 1}$$

in S^1 are such that $0 < \alpha < \beta \leq 2\pi$, and $[\beta\,,\,\alpha + 2\pi] \cup [\alpha + 2\,(m+1)\,\pi\,,\,\beta + 2\,(m+1)\,\pi]$ forms an H^2-wavelet set.

6. MRA associated Three-Interval Wavelet Sets

In this Section, we consider three-interval wavelet sets and determine if these are associated with a multiresolution analysis. In [9], it has been shown that if p is an odd natural number, then the wavelet set $W\,(j, p)$ is not associated with an MRA, for $j \geq 2$. Using a result stated below obtained in [8], we provide an alternative proof of this result. In addition, we prove that if $p = 2^j - 2$, then the wavelet set $W\,(j, p)$ is associated with an MRA, where $j \geq 2$. That each of the two-interval wavelet sets is associated with a multiresolution analysis can be settled on the same lines.

Theorem 6.1. [8] *For an MSF wavelet ψ with $|\hat{\psi}| = \chi_W$, the following are equivalent:*

(i) ψ is associated with an MRA.

(ii) $\mu(W^S \cap (W^S + 2k\pi)) = 2\pi\delta_{k,0}\,,\ k \in \mathbb{Z}$.

(iii) $\left\{W^S + 2k\pi\ :\ k \in \mathbb{Z}\right\}$ forms a partition of $\mathbb{R}$,

where $W^S = \bigcup_{n=1}^{\infty} 2^{-n} W$.

Consider the three-interval wavelet set

$$W(j,p) = \left[-2\left(1-\frac{2p+1}{2^{j+1}-1}\right)\pi\,,\,-\left(1-\frac{2p+1}{2^{j+1}-1}\right)\pi\right] \cup \left[\frac{2(p+1)\pi}{2^{j+1}-1}\,,\,\frac{2(2p+1)\pi}{2^{j+1}-1}\right]$$
$$\cup\left[\frac{2^{j+1}(2p+1)\pi}{2^{j+1}-1}\,,\,\frac{2^{j+2}(p+1)\pi}{2^{j+1}-1}\right]$$

denoted by W, for simplicity. Then

$$W^S = \left[-\left(1-\frac{2p+1}{2^{j+1}-1}\right)\pi\,,\,0\right) \cup \left(0\,,\,\frac{2(p+1)\pi}{2^{j+1}-1}\right]$$
$$\cup\left(\bigcup_{k=1}^{j} 2^{j+1-k}\left[\frac{(2p+1)\pi}{2^{j+1}-1}\,,\,\frac{2(p+1)\pi}{2^{j+1}-1}\right]\right),$$

where $j \geq 2$ and $1 \leq p \leq 2^j - 2$.

Theorem 6.2. *The wavelet set $W(j,p)$, where p is odd and $j \geq 2$ is not associated with a multiresolution analysis.*
Proof. Since

$$\frac{2^j(2p+1)\pi}{2^{j+1}-1} < (p+1)\pi < \frac{2^{j+1}(p+1)\pi}{2^{j+1}-1}$$

for $j \geq 2$ and $1 \leq p \leq 2^j - 2$, we have

$$(p+1)\pi \in 2^{j+1-k}\left[\frac{(2p+1)\pi}{2^{j+1}-1}\,,\,\frac{2(p+1)\pi}{2^{j+1}-1}\right],$$

when $k = 1$. Therefore, the 2π-translation map $\tau : W^S \longrightarrow [0\,,\,2\pi)$ cannot be a bijection, if p is odd. It is because, when $p = 2m+1$, where $m = 0,\ 1,\ \ldots,\ \left(2^{j-1}-2\right)$,

$$\tau\left(\left[(p+1)\pi\,,\,\frac{2^{j+1}(p+1)\pi}{2^{j+1}-1}\right]\right) = \left[(p+1)\pi\,,\,\frac{2^{j+1}(p+1)\pi}{2^{j+1}-1}\right] - (p+1)\pi$$

intersects

$$\tau\left(\left(0\,,\,\frac{2(p+1)\pi}{2^{j+1}-1}\right]\right) = \left(0\,,\,\frac{2(p+1)\pi}{2^{j+1}-1}\right].$$

Thus by (iii) of Theorem 6.1, $W(j,p)$ is not associated with a multiresolution analysis. □

Theorem 6.3. *The wavelet set $W(j,p)$, where $p = 2^j - 2$ and $j \geq 2$, is associated with a multiresolution analysis.*
Proof. Writing W for $W(j, 2^j-2)$, we have

$$W^S = \left[-\frac{2\pi}{2^{j+1}-1}\,,\,0\right) \cup \left(0\,,\,\left(\frac{2^{j+1}-2}{2^{j+1}-1}\right)\pi\right] \cup \bigcup_{k=1}^{j} 2^k\left[\left(\frac{2^{j+1}-3}{2^{j+1}-1}\right)\pi\,,\,\left(\frac{2^{j+1}-2}{2^{j+1}-1}\right)\pi\right].$$

Since the map $\tau : W^S \longrightarrow [0\,,\,2\pi)$ defined by

$$\tau(x) = \begin{cases} x + 2\pi, & \text{if } x \in \left[-\frac{2\pi}{2^{j+1}-1}\,,\,0\right) \\ x, & \text{if } x \in \left(0\,,\,\left(\frac{2^{j+1}-2}{2^{j+1}-1}\right)\pi\right] \cup 2\left[\left(\frac{2^{j+1}-3}{2^{j+1}-1}\right)\pi\,,\,\left(\frac{2^{j+1}-2}{2^{j+1}-1}\right)\pi\right] \\ x - (2^m - 2)\pi, & \text{if } x \in 2^m\left[\left(\frac{2^{j+1}-3}{2^{j+1}-1}\right)\pi\,,\,\left(\frac{2^{j+1}-2}{2^{j+1}-1}\right)\pi\right] \end{cases}$$

where $m = 2,\ 3,\ \ldots,\ j$, is a bijection, $\left\{W^S + 2k\pi : k \in \mathbb{Z}\right\}$ forms a partition of $\mathbb{R}$. The proof follows from (iii) of Theorem 6.1. □

Remark 6.1. It can be worked out manually that when j is 3 or 5, there is no three-interval wavelet set which is associated with an MRA except when $p = 2^j - 2$. In case j is 7, $W(7, 36)$ and $W(7, 90)$ are the other two three-interval wavelet sets associated with a multiresolution analysis in addition to that of $W\left(7, 2^7 - 2\right)$.

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