Riemann solution for ideal isentropic magnetogasdynamics

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Outline

Basic equations

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (p + \rho u^2 + B^2/2)_x &= 0, \\ B_t + (Bu)_x &= 0. \end{aligned} \tag{1}$$

Riemann problem

$$\frac{\partial V}{\partial t} + \frac{\partial F(V)}{\partial x} = 0, \tag{2}$$

where $V = (\rho, \rho u, B)^{tr}, F(V) = (\rho u, \rho u^2 + p + \frac{B^2}{2}, Bu)^{tr}$.
With piecewise constant initial data

$$V(x,0) = \begin{cases} V_l, & x < 0, \\ V_r, & x > 0. \end{cases} \tag{3}$$

Riemann problem

Riemann invariants

$$\prod_1^1 = \frac{B}{\rho}, \qquad \prod_2^1 = u + \int^\rho \frac{w(\theta)}{\theta} d\theta, \tag{4}$$

$$\prod_1^2 = u, \qquad \prod_2^2 = p + \frac{B^2}{2}, \tag{5}$$

$$\prod_1^3 = \frac{B}{\rho}, \qquad \prod_2^3 = u - \int^\rho \frac{w(\theta)}{\theta} d\theta. \tag{6}$$

Riemann problem

Shock waves

(i) Rankine-Hugoniot jump conditions:

$$F(V) - F(V_l) = s(V - V_l), \tag{7}$$

(ii) Lax entropy conditions:

$$\lambda_{i-1}(V_l) < s_i < \lambda_i(V_l), \quad \lambda_i(V) < s_i < \lambda_{i+1}(V). \ \ i = 1, 3. \tag{8}$$

Riemann problem

Theorem Across 1-shock (respectively, 3-shock), $\rho > \rho_l$, $u < u_l$, $p > p_l$ and $B > B_l$ (respectively, $\rho < \rho_l$, $u < u_l$, $p < p_l$ and $B < B_l$) if and only if, the Lax conditions hold.

Riemann problem

Introducing the variable $\tilde{u} = u - s$ become

$$\rho \tilde{u} = \rho_l \tilde{u}_l, \tag{9}$$

$$p + \rho \tilde{u}^2 + \frac{B^2}{2} = p_l + \rho_l {\tilde{u}_l}^2 + \frac{B_l^2}{2}, \tag{10}$$

$$B \tilde{u} = B_l \tilde{u}_l. \tag{11}$$

Riemann problem

Using (9)-(11)

$$B = \left(\frac{B_l}{\rho_l}\right)\rho, \tag{12}$$

$$\tilde{u}_l^2 = \frac{(p_l + B_l^2/2 - p - B^2/2)\rho/\rho_l}{\rho_l - \rho}, \tag{13}$$

$$s_1 = u_l - \sqrt{\frac{(p_l + B_l^2/2 - p - B^2/2)\rho/\rho_l}{\rho_l - \rho}}, \tag{14}$$

$$u = \frac{\rho_l u_l}{\rho} + \left(\frac{\rho - \rho_l}{\rho}\right)s_1. \tag{15}$$

Riemann problem

$$B = \left(\frac{B_r}{\rho_r}\right)\rho, \tag{16}$$

$$s_3 = u_r + \sqrt{\frac{(p_r + B_r^2/2 - p - B^2/2)\rho/\rho_r}{\rho_r - \rho}}, \tag{17}$$

$$u = \frac{\rho_r u_r}{\rho} + \left(\frac{\rho - \rho_r}{\rho}\right) s_3, \tag{18}$$

where s_3 is the speed of 3-shock wave.

Riemann problem

Contact discontinuities

$$\lambda_2(V_l) = \lambda_2(V) = s_2. \tag{19}$$

From (19), we get $u = u_l$, and using (19) in (10), we obtain

$$p + \frac{B^2}{2} = p_l + \frac{B_l^2}{2}, \tag{20}$$

so u and $p + \frac{B^2}{2}$ are constant across the contact discontinuity.

Riemann problem

Rarefaction waves

(i) Riemann invariants are constant across the wave

$$B = \left(\frac{B_l}{\rho_l}\right)\rho, \tag{21}$$

$$u = u_l + \int_{\rho}^{\rho_l} \frac{w(\theta)}{\theta} d\theta. \tag{22}$$

and (ii) divergence of characteristics

$$\lambda_i(V_l) < \lambda_i(V), i = 1, 3. \tag{23}$$

Riemann problem

Theorem Across 1-rarefaction waves (respectively, 3-rarefaction waves), $\rho < \rho_l$, $u > u_l, B < B_l$ and $p < p_l$ (respectively, $\rho > \rho_l$, $u > u_l, B > B_l$ and $p > p_l$) if and only if, the characteristic speed increases from left hand to right hand state.

Solution Strategy

The three waves separate four constant states, which from the left to the right are $Z_1 = Z_l, Z_2, Z_3$, and $Z_4 = Z_r$, where $Z = (\rho, u, B)^{tr}$. The main physical quantities to be determined are $s_1, s_3, \rho_2, \rho_3, B_2, B_3$ and u_2.

Solution Strategy

Physical quantities across left shock

$$u_2 = u_1 - f_l(\rho_2, Z_1), \tag{24}$$

with $f_l(\rho_2, Z_1) = \sqrt{A_1\rho_2^2 + A_2\rho_2^\gamma + A_3\rho_2^{\gamma-1} + A_4\rho_2 + \frac{A_5}{\rho_2} + A_6}$,
where $A_1 = \frac{B_1^2}{2\rho_1^3}$, $A_2 = \frac{A}{\rho_1}$, $A_3 = -A$, $A_4 = -\frac{B_1^2}{2\rho_1^2}$, $A_5 = \left(A\rho_1^\gamma + \frac{B_1^2}{2}\right)$, $A_6 = \left(-A\rho_1^{\gamma-1} - \frac{B_1^2}{2\rho_1}\right)$.

$$p_2 + \frac{B_2^2}{2} = g_l = \left(\frac{B_1^2}{2\rho_1^2}\right)\rho_2^2 + A\rho_2^\gamma = D_1\rho_2^2 + A\rho_2^\gamma, \tag{25}$$

where $D_1 = \left(\frac{B_1^2}{2\rho_1^2}\right)$.

Solution Strategy

Physical quantities across left rarefaction wave

$$u_2 = u_1 - f_l(\rho_2, Z_1), \tag{26}$$

where $f_l(\rho_2, Z_1) = \int_{\rho_1}^{\rho_2} \frac{w(\theta)}{\theta} d\theta$.

$$p_2 + \frac{B_2^2}{2} = g_l = \left(\frac{B_1^2}{2\rho_1^2}\right)\rho_2^2 + A\rho_2^\gamma = D_1\rho_2^2 + A\rho_2^\gamma. \tag{27}$$

Solution Strategy

Physical quantities across right shock wave

$$u_3 = u_4 + f_r(\rho_3, Z_4), \tag{28}$$

with $f_r(\rho_3, Z_4) = \sqrt{C_1\rho_3^2 + C_2\rho_3^\gamma + A_3\rho_3^{\gamma-1} + C_3\rho_3 + \frac{C_4}{\rho_3} + C_5}$,
where $C_1 = \frac{B_4^2}{2\rho_4^3}$, $C_2 = \frac{A}{\rho_4}$, $C_3 = -\frac{B_4^2}{2\rho_4^2}$, $C_4 = \left(A\rho_4^\gamma + \frac{B_4^2}{2}\right)$, $C_5 = \left(-A\rho_4^{\gamma-1} - \frac{B_4^2}{2\rho_4}\right)$.

$$p_3 + \frac{B_3^2}{2} = g_r = \left(\frac{B_4^2}{2\rho_4^2}\right)\rho_3^2 + A\rho_3^\gamma = D_2\rho_3^2 + A\rho_3^\gamma, \tag{29}$$

where $D_2 = \left(\frac{B_4^2}{2\rho_4^2}\right)$.

Solution Strategy

Physical quantities across right rarefaction wave

$$u_3 = u_4 + f_r(\rho_3, Z_4), \tag{30}$$

where $f_r(\rho_3, Z_4) = \int_{\rho_4}^{\rho_3} \frac{w(\theta)}{\theta} d\theta$.

$$p_3 + \frac{B_3^2}{2} = g_r = \left(\frac{B_4^2}{2\rho_4^2}\right)\rho_3^2 + A\rho_3^\gamma = D_2\rho_3^2 + A\rho_3^\gamma. \tag{31}$$

Solution Strategy

$$f(\rho_2, \rho_3, Z_1, Z_4) \equiv f_l + f_r + \Delta u = 0, \tag{32}$$

where $\Delta u = u_4 - u_1$, and the functions f_l and f_r are given by

$$f_l(\rho_2, Z_1) = \begin{cases} \sqrt{A_1\rho_2^2 + A_2\rho_2^{\gamma} + A_3\rho_2^{\gamma-1} + A_4\rho_2 + \frac{A_5}{\rho_2} + A_6}, & \text{if } \rho_1 < \rho_2, \\ \int_{\rho_1}^{\rho_2} \frac{w(\theta)}{\theta} d\theta & \text{if } \rho_1 \geq \rho_2, \end{cases}$$

$$f_r(\rho_3, Z_4) = \begin{cases} \sqrt{C_1\rho_3^2 + C_2\rho_3^{\gamma} + A_3\rho_3^{\gamma-1} + C_3\rho_3 + \frac{C_4}{\rho_3} + C_5} & \text{if } \rho_4 < \rho_3, \\ \int_{\rho_4}^{\rho_3} \frac{w(\theta)}{\theta} d\theta & \text{if } \rho_4 \geq \rho_3. \end{cases}$$

Solution Strategy

$$g(\rho_2, \rho_3, Z_1, Z_4) \equiv g_r(\rho_3, Z_4) - g_l(\rho_2, Z_1) = 0, \tag{33}$$

where the functions g_l and g_r are given by

$$g_l(\rho_2, Z_1) = D_1 \rho_2^2 + A\rho_2^\gamma, \tag{34}$$

$$g_r(\rho_3, Z_4) = D_2 \rho_3^2 + A\rho_3^\gamma. \tag{35}$$

Numerical solution

Numerical Examples

Test	ρ_l	u_l	B_l	ρ_r	u_r	B_r
1	6.0	0.0	1.0	1.0	0.0	0.1
2	1.0	-0.5	0.4	1.0	1.5	0.4
3	5.99924	19.5975	100.894	7.99242	-6.19633	46.0950
4	0.96	1.0833	2.8333	1.7741	1.1187	4.0

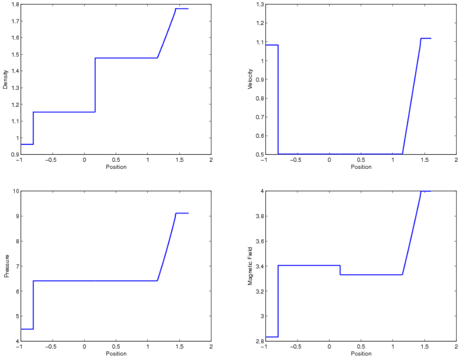
Test	ρ_2	u_2	B_2	ρ_3	B_3
1	2.72388	1.02259	0.45398	2.78509	0.27850
2	0.33228	0.4999	0.1329	0.33228	.1329
3	6.8226	14.11506	114.74116	19.85648	114.51907
4	1.15422	0.50209	3.4065	1.47762	3.33155

Numerical solution

No Roots

The nonlinear equations (32) and (33) are not solvable If Δu is very large; for instance, for the data
$\rho_1 = 1.0, u_1 = -2.0, B_1 = 0.4, \rho_4 = 1.0, u_4 = 2.0$ and $B_4 = 0.4$, system of algebraic equations is not solvable.

Numerical solution



References

1. J. Smoller, *Shock waves and reaction-diffusion equations*, Springer-Verlag, New York, 1994.
2. E. F. Toro, *Riemann solvers and numerical methods for fluid dynamics*, Springer-Verlag, Berlin, 2nd ed. 1997.
3. T. Raja Sekhar and V.D. Sharma, *Riemann problem and elementary wave interactions in isentropic magnetogasdynamics*, Nonlinear Analysis: Real World Applications 11 (2010), no. 2, 619-636.
4. Y. Liu and W. Sun, *Riemann problem and wave interactions in magnetogasdynamics*, J. Math. Anal. Appl. 397 (2013), no. 2, 454-466.
5. L. Quartapelle, L. Castelletti, A. Guardone and G. Quaranta, Solution of the Riemann problem of classical gasdynamics, J. Comput. Phys 190 (2003), 118-140.

Thank You.