Self similar solutions in shallow water equations

T. Raja Sekhar

Department of Mathematics
NIT Rourkela

March 14, 2012
Outline

1 Introduction
   - Infinitesimal transformations
   - Infinitesimal generator

2 Problem analysis
   - Power-series method
   - Compatible condition
   - Usage of similarity variables
Infinitesimal transformations: Consider a one-parameter Lie group of transformations \( x^* = X(x; \epsilon) \) with identity \( \epsilon = 0 \) and law of composition \( \phi \). If we expand \( x^* = X(x; \epsilon) \) about \( \epsilon = 0 \) we get

\[
x^* = x + \epsilon \left( \frac{\partial X}{\partial \epsilon} \right)_{\epsilon=0} + \frac{\epsilon^2}{2} \left( \frac{\partial^2 X}{\partial \epsilon^2} \right)_{\epsilon=0} + \cdots
\]

\[
x^* = x + \epsilon \xi(x) + O(\epsilon^2)
\]

where \( \xi(x) = \left( \frac{\partial X}{\partial \epsilon} \right)_{\epsilon=0} \). This is called infinitesimal transformation of \( x^* = X(x; \epsilon) \) and the components of \( \xi(x) \) are called infinitesimals of \( x^* = X(x; \epsilon) \).
**Infinitesimal generator:** The infinitesimal generator of the one-parameter Lie group of transformations $x^* = X(x; \epsilon)$ is the operator

$$X = X(x) = \xi(x). \nabla = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}$$

where

$$\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \cdots, \frac{\partial}{\partial x_n} \right)$$

For any differentiable function $F(x) = F(x_1, x_2, x_3, \cdots, x_n)$

$$XF(x) = \xi(x). \nabla F(x) = \sum_{i=1}^{n} \xi_i(x) \frac{\partial F(x)}{\partial x_i}$$
Problem analysis

We consider the system of equations which governs the one dimensional modified shallow water equations as follows [?]

\[
\begin{align*}
    h_t + h u_x + u h_x &= 0, \\
    u_t + \frac{g(h + H)}{h} h_x + uu_x &= 0, \\
\end{align*}
\] (1)

where \(x, t\) are the independent variables denoting the space and time respectively and

- \(u = x\)-component of fluid velocity,
- \(h = \) variable depth
- \(g = \) acceleration due to gravity, \(H = \frac{k_0}{g}\).
Firstly, we consider Lie group of transformations with independent variables $x,t$ and dependent variables $u, h$ for the problem

\begin{align*}
\tilde{x} &= \tilde{x}(x, t, h, u; \epsilon), \\
\tilde{t} &= \tilde{t}(x, t, h, u; \epsilon), \\
\tilde{u} &= \tilde{u}(x, t, h, u; \epsilon), \\
\tilde{h} &= \tilde{h}(x, t, h, u; \epsilon).
\end{align*}

(2)
where $\epsilon$ is the group parameter. The infinitesimal generator of the group (2) can be expressed in the following vector form

$$V = \xi^x \frac{\partial}{\partial x} + \xi^t \frac{\partial}{\partial t} + \eta^u \frac{\partial}{\partial u} + \eta^h \frac{\partial}{\partial h}$$

in which $\xi^x$, $\xi^t$, $\eta^u$, $\eta^h$ are infinitesimal functions of the group variables. Then the corresponding one-parameter Lie group of transformations is given by
\[ \tilde{x} = x + \epsilon \xi^x(x, t, h, u) + O(\epsilon^2), \]
\[ \tilde{t} = t + \epsilon \xi^t(x, t, h, u) + O(\epsilon^2), \]
\[ \tilde{u} = u + \epsilon \eta^u(x, t, h, u) + O(\epsilon^2), \]
\[ \tilde{h} = h + \epsilon \eta^h(x, t, h, u) + O(\epsilon^2). \]

Since the system of one-layer shallow-water equations has at most first-order derivatives, the first prolongation of the generator should be considered in the form:
Pr' V = V + \tau^u_x \frac{\partial}{\partial u_x} + \tau^u_t \frac{\partial}{\partial u_t} + \tau^h_x \frac{\partial}{\partial h_x} + \tau^h_t \frac{\partial}{\partial h_t} \tag{3}

where
\tau^u_t = \eta^u_t + \eta^u_t u_t + \eta^h_t h_t - u_x(\xi^x_t + \xi^x_u u_t + \xi^x_h h_t) - u_t(\xi^t_t + \xi^t_u u_t + \xi^t_h h_t)
\tau^u_x = \eta^u_x + \eta^u_x u_x + \eta^h_x h_x - u_x(\xi^x_x + \xi^x_u u_x + \xi^x_h h_x) - u_t(\xi^t_x + \xi^t_u u_x + \xi^t_h h_x)
\tau^h_t = \eta^h_t + \eta^h_t u_t + \eta^h_t h_t - h_x(\xi^x_t + \xi^x_u u_t + \xi^x_h h_t) - h_t(\xi^t_t + \xi^t_u u_t + \xi^t_h h_t)
\tau^h_x = \eta^h_x + \eta^h_x u_x + \eta^h_x h_x - h_x(\xi^x_x + \xi^x_u u_x + \xi^x_h h_x) - h_t(\xi^t_x + \xi^t_u u_x + \xi^t_h h_x).
if we apply the first prolongation of the infinitesimal generator (3) to the system of partial differential equations (1)

\[
Pr' V(h_t + hu_x + uh_x)h_t = -uh_x - hu_x = 0,
\]
\[
Pr' V(u_t + \frac{g(h + H)}{h} h_x + uu_x)u_t = -uu_x - \frac{g(h + H)}{h} h_x = 0.
\]

then we obtained the following system of equations

\[
\eta^u h_x + \eta^h u_x + \tau^h_t + u\tau^h_x + h\tau^u_x = 0,
\]
\[
-\frac{gH}{h^2} \eta^h + \eta^u u_x + g\tau^h_x + \tau^u_t + u\tau^u_x = 0.
\]

which gives us the following determining equations
Firstly, we choose the first order of power-series of the infinitesimals which are given by

\[
\begin{align*}
\xi^x &= a_0 + a_1 x + a_2 t + a_3 h + a_4 u \\
\xi^t &= b_0 + b_1 x + b_2 t + b_3 h + b_4 u \\
\eta^u &= c_0 + c_1 x + c_2 t + c_3 h + c_4 u \\
\eta^h &= d_0 + d_1 x + d_2 t + d_3 h + d_4 u
\end{align*}
\]

where \(a_i, b_i, c_i, a_i, (i = 0, 1, 2, 3, 4)\) are constant coefficients.
Then substituting these power-series forms into the determining equations and straightforward calculations for the first order of power-series forms, we find three-parameter Lie group of transformations of one-layer shallow-water equations as follows

\[\begin{align*}
\xi^t &= a_1 t + a_4, \\
\xi^x &= a_1 x + a_2 t + a_3, \\
\eta^u &= a_2, \\
\eta^h &= 0.
\end{align*}\]
These transformations provide the following three Lie point generators:

\[ X_1 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \]
\[ X_2 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \]
\[ X_3 = \frac{\partial}{\partial x}. \]

For \( a_1 \neq 0, b_2 \neq 0 \) and \( b_0 \neq 0 \) respectively.
Consider the infinitesimal generators $V_A$, $V_B$ defined by

$$V_A = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3$$
$$= \alpha_1 t \frac{\partial}{\partial t} + (\alpha_1 x + \alpha_2 t + \alpha_3) \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial u},$$

and

$$V_B = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$
$$= \beta_1 t \frac{\partial}{\partial t} + (\beta_1 x + \beta_2 t + \beta_3) \frac{\partial}{\partial x} + \beta_2 \frac{\partial}{\partial u},$$

$\alpha_i, \beta_i \in \mathbb{R}$. 

T. Raja Sekhar, NIT, Rourkela
Compatible Condition

For the compatible condition we consider the following relation

\[ [V_A, V_B] = V_A V_B - V_B V_A = 0 \]

which yields

\[ \alpha_3 \beta_1 - \alpha_1 \beta_3 = 0. \]
Since the system is invariant under the group generated by $V_A$, we introduce a set of canonical variables defined by,

$$V_A \bar{\tau} = 1, \quad V_A \bar{\xi} = 0, \quad V_A \bar{U} = 0, \quad V_A \bar{P} = 0,$$

allowing one to express $V_A$ as a translation with respect to $\bar{\tau}$, the characteristic conditions are

$$\frac{dt}{\alpha_1 t} = \frac{dx}{\left( \alpha_1 x + \alpha_2 t + \alpha_3 \right)} = \frac{du}{\alpha_2} = \frac{d\bar{\tau}}{1}, \quad \text{(4)}$$
where $\alpha_1$, $\alpha_2$, and $\alpha_3$ are non-zero constants. Hence equation (4) yield the following transformation of variables

$$
\bar{\tau} = \frac{1}{\alpha_1} \ln t,
$$

$$
\bar{\xi} = t^{-1} e^{\frac{(\alpha_1 x + \alpha_3)}{\alpha_2 t}},
$$

$$
\bar{U} = e^{\mu t} \alpha_1^{-1},
$$

$$
\bar{P} = h.
$$
Now we can express $V_B$ using the new variables as

$$
\bar{V}_B = V_B \bar{\tau} \frac{\partial}{\partial \bar{\tau}} + V_B \bar{\xi} \frac{\partial}{\partial \bar{\xi}} + V_B \bar{U} \frac{\partial}{\partial \bar{U}} + V_B \bar{P} \frac{\partial}{\partial \bar{P}},
$$

$$
= \frac{\beta_1}{\alpha_1} \frac{\partial}{\partial \bar{\tau}} + \frac{\alpha_1 \beta_2 - \alpha_2 \beta_1}{\alpha_1} \bar{\xi} \frac{\partial}{\partial \bar{\xi}} + \frac{\alpha_1 \beta_2 - \alpha_2 \beta_1}{\alpha_1} \bar{U} \frac{\partial}{\partial \bar{U}}.
$$

In a similar manner, we choose a second set of canonical variables allowing $\bar{V}_B$ to be written as translation with respect to $\xi$, i.e.,
\[ \bar{V}_B \tau = 0, \quad \bar{V}_B \xi = 1, \quad \bar{V}_B U = 0, \quad \bar{V}_B P = 0. \quad (5) \]

The characteristic conditions associated with (5) yield the following transformation of variables

\[ \frac{\alpha_1 d\bar{\tau}}{\beta_1} = \frac{\alpha_1 d\bar{\xi}}{(\alpha_1 \beta_2 - \alpha_2 \beta_1)\bar{\xi}} = \frac{\alpha_1 d\bar{U}}{(\alpha_1 \beta_2 - \alpha_2 \beta_1)\bar{U}} = \frac{d\xi}{1}. \]

The characteristic conditions yield the following transformation of variables
\[ \tau = \ln t \frac{\alpha_2 \beta_1 K + 1}{\alpha_1} - \frac{\beta_1 K (x + \frac{\alpha_3}{\alpha_1})}{t}, \quad (6) \]

\[ \xi = \ln t^{-\alpha_2 K} + \frac{\alpha_1 K (x + \frac{\alpha_3}{\alpha_1})}{t}, \quad (7) \]

\[ u = \ln U + \frac{(x + \frac{\alpha_3}{\alpha_1})}{t}, \quad (8) \]

\[ h = P, \quad (9) \]

where \( K = \frac{1}{(\alpha_1 \beta_2 - \alpha_2 \beta_1)}. \)
Using the above transformation in the governing system (1), we get the following system of PDEs:

\[
\left(\frac{\alpha_2 \beta_1 K + 1}{\alpha_1} - \beta_1 K \ln U\right) \frac{\partial P}{\partial \tau} + \left(\alpha_1 K \ln U - \alpha_2 K\right) \frac{\partial P}{\partial \xi} - \beta_1 K \frac{P}{P+H} U \frac{\partial U}{\partial \tau} + \frac{\beta_1 Kg(P+H)}{P} U \frac{\partial P}{\partial \tau} + \frac{\alpha_1 Kg(P+H)}{P} U \frac{\partial P}{\partial \xi} + U \ln U = 0.
\]
By considering $U = 1$, the above system of PDEs can be reduced as follows

\[- \beta_1 \frac{\partial P}{\partial \tau} + \alpha_1 \frac{\partial P}{\partial \xi} = 0, \]
\[\frac{\partial P}{\partial \xi} - \beta_1 \frac{\partial P}{\partial \tau} = 0. \tag{11}\]

Equation (11) can be solved as

\[P(\xi, \tau) = P_1(\eta) \tag{12}\]
where $\eta = \tau + \frac{\beta_1}{\alpha_1} \xi$. Using (12) in the equation (10), we obtained
\[
\frac{dP_1}{d\eta} + \alpha_1 P_1 = 0 \tag{13}
\]
equation (13) can be solved
\[
P_1 = Ce^{-\alpha_1 \eta} \tag{14}
\]
where $C$ is an arbitrary constant and thus, in view of the equations (6), (12) and (14) the solution of the system (1) can be expressed as follows

$$h = \frac{C}{t}, \quad u = \frac{x + \frac{\alpha_3}{\alpha_1}}{t}.$$  

(15)
Reference


Thank You.