Dynamics of multilayered viscoelastic beams

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Abstract. Viscoelastic materials store as well as dissipate energy to the thermal domain under deformation. Two efficient modelling techniques reported in literature use coupled (thermo-mechanical) ATF (Augmenting Thermodynamic Fields) displacements and ADF (Anelastic Displacement Fields) displacements, to represent the constitutive relationship in time domain by using certain viscoelastic parameters. Viscoelastic parameters are first extracted from the storage modulus and loss factor normally reported in handbooks with the help of Genetic Algorithm and then constitutive relationships are used to obtain the equations of motion of the continuum after discretizing it with finite beam elements. The equations of motion are solved to get the frequency response function and modal damping ratio. The process may be applied to study the dynamic behaviour of composite beams and rotors comprising of several viscoelastic layers. Dynamic behaviour of a composite beam, formed by concentric layers of steel and aluminium is studied as an example.

Keywords: viscoelastic beam; augmenting thermodynamic field; anelastic displacement field; viscoelastic model parameters; modal damping ratio; composite beam.

1. Introduction

A viscoelastic material stores energy and dissipates it in the thermal domain when subjected to dynamic loading and most interestingly the storage and loss of energy depends upon the frequency of excitation. To find a time domain model to represent the behaviour of viscoelastic solids found
interest of many researchers. Bagley and Torvik (1983, 1985) represented frequency dependent
behaviour of viscoelastic solids by using four-model parameters and differential operators of
fractional order. The time-domain model with ordinary integer differential operators was developed
by Golla and Hughes (1985), who incorporated the hereditary integral form of the viscoelastic
constitutive law in a finite element model. The finite-element equations are derived in the Laplace
domain through the Ritz technique. McTavish and Hughes (1992, 1993) extended the Golla-Hughes
model and formulated the GHM (Golla-Hughes-McTavish) model for linear viscoelastic structures.
In this formulation, the material modulus is modelled as a collection of mini-oscillators. Lesieutre
(1989) developed another time-domain model, using ‘Augmenting Thermodynamic Fields (ATF)’
approach to model frequency-dependent material damping of linear viscoelastic structures in a finite
element context. The procedure introduces a thermal coordinate to take into account the dissipation.
Lesieutre and Mingori (1990) and later Lesieutre (1992) developed a one-dimensional formulation
of the ATF model. In a recent paper, Roy et al. (2008) used the ATF approach to model a
viscoelastic continuum of a rotor shaft, to obtain the equations of motion and studied the dynamic
behaviour in terms of stability limit speed and unbalance response. The ‘Anelastic Displacement
Field’ ADF, (Lesieutre and Bianchini (1995), Lesieutre et al. (1996)) approach was developed to
extend the ATF method to three-dimensional states. In ADF approach, the displacement field was
composed of an elastic component and an anelastic component, where the anelastic field is
introduced to take into account the dissipation. All the approaches, ATF, ADF and GHM thus
employ additional co-ordinates to model damping more accurately. Whereas the ‘dissipation co-
ordinate’ of GHM is internal to individual elements, it is continuous for ATF and ADF approaches
from element to element. For this advantage ATF and ADF approaches are used in this work to
represent the viscoelastic material behaviour.

Viscoelastic parameters used in these approaches are determined initially from the storage
modulus and loss factor reported by Lazan (1968). For this a Genetic Algorithm based approach is
adopted to minimize the error between the hysteresis loops predicted by ATF, ADF approaches each
and the same predicted by the complex modulus, for a viscoelastic solid bar under sinusoidally
varying axial load. Subsequently the extracted ATF and ADF parameters are used to represent the
constitutive equations and obtain the equations of motion for the continuum of a generally
multilayered viscoelastic beam, discretized using finite beam elements. These equations of motion
are studied for the dynamics of the viscoelastic composite beam.

For an example the dynamics of a cantilevered composite beam made of two concentric layers of
Steel and Aluminium is studied, where behaviours of both steel and aluminium are represented by
viscoelastic models. It has been observed that proper choice of placement of layers and their
thickness ratios help in postponing the first natural frequency, increasing modal damping leading to
reduction in vibration response.

2. Constitutive relations and equations of motion

Equations of motion of a beam made of single viscoelastic material have been given by Lesieutre
and Mingori (1990) by using ATF approach. For the purpose of writing the equations of motion
using the ADF approach, as it is not in the literature, as well as for the sake of completeness, this
section is written as an extension of the work by Lesieutre and Mingori (1990) and Lesieutre et al.
(1996).
2.1 Constitutive relationships

In this section a beam made of several viscoelastic layers has been assumed and correspondingly the equations of motion have been derived by using ATF and ADF approaches. All the layers are assumed to be perfectly bonded and no slip takes place between the layers during bending. Fig. 1 shows the sectional view of a multi-layered beam. The Helmholtz free energy for ATF and ADF approaches of \( j \)th layers are given by Eqs. (1a) and (1b) respectively. Corresponding material constitutive relations are given by Eqs. (2a) and (2b).

\[
H_j = \frac{1}{2} (E_j \varepsilon_j^2 - 2 \delta_j \varepsilon_j \dot{\xi}_j + \alpha_j \dot{\xi}_j^2) \tag{1a}
\]

\[
H_j = \frac{1}{2} (E_j \varepsilon_j^2 - 2E_j \varepsilon_j^A + E_j^A \dot{e}_j^A) \tag{1b}
\]

\[
\sigma_j = \frac{\partial H_j}{\partial \varepsilon_j} = (E_j \varepsilon_j - \delta_j \dot{\xi}_j); \quad \dot{\varepsilon}_j = \frac{\partial H_j}{\partial \dot{\xi}_j} = \delta_j \varepsilon_j - \alpha_j \dot{\xi}_j \tag{2a}
\]

\[
\sigma_j = \frac{\partial H_j}{\partial \varepsilon_j^A} = E_j (\varepsilon_j - \varepsilon_j^A); \quad \dot{\varepsilon}_j^A = \frac{\partial H_j}{\partial \dot{e}_j^A} = E_j^A (\varepsilon_j - \varepsilon_j^A) \tag{2b}
\]

where the subscript ‘\( j \)’ stands for the contribution obtained from the \( j \)th layer.

In ATF approach \( E \) is the elastic modulus, \( \sigma \) is the mechanical stress, \( \varepsilon \) is the mechanical strain, \( \dot{\xi} \) is the augmenting thermodynamic field (ATF), \( \mathcal{A} \) is the affinity, \( \alpha \) is a material property relating the changes in \( \mathcal{A} \) to \( \dot{\xi} \) and \( \delta \) is the strength of coupling between the mechanical displacement field and the thermodynamic field.

In the case of ADF approach, \( E \) is the elastic modulus, \( \sigma \) is the total stress, \( \varepsilon \) is the total strain, \( \sigma^A \) is the anelastic stress, \( \varepsilon^A \) is the anelastic strain, \( E^A \) is the anelastic material property and \( c = E^A/E \) is a constant.

2.2 Equations of motion of beam made of multiple viscoelastic layers

A uniform, plane, slender beam made of several viscoelastic materials arranged in layers, is considered. When the beam is deformed, it is assumed that a transverse section, originally plane, will remain plane. The stress normal to the transverse plane (parallel to \( x \)-axis) is assumed to be the main stress. The \( x \)-axis and \( y \)-axis are defined to be parallel and perpendicular to the beam axis,
respectively with corresponding displacement components \( u(x,y) \) and \( v(x,y) \). To this end the following approximations are used.

\[
v(x,y) = v(x) , \quad u(x,y) = -yv'(x) \quad (3)
\]

Using the constitutive relations, the equations of motions are obtained as a set of two coupled partial differential equations. The equation of evolution for the mechanical displacement field is developed from the consideration of momentum balance or by assuming zero body force. Assuming the time rate of change of dissipation field is proportional to thermal stress or proportional to its deviation from an equilibrium value, a first order differential equation, i.e., a relaxation equation is obtained

For ATF approach

\[
\dot{\xi}_j = S \dot{\delta}_j = -B_j (\dot{\xi}_j - \overline{\xi}_j) \quad (4a)
\]

and for ADF approach

\[
\dot{\varepsilon}_j = S \sigma_j = -b_j (\varepsilon_j - \overline{\varepsilon}_j) \quad (4b)
\]

In the preceding equation \( B_j \) or \( b_j \) is the inverse of relaxation time for the \( j \)th layer, \( S \) is a constant of proportionality, and \( \overline{\xi}_j, \overline{\varepsilon}_j \) denote respectively the values of \( \xi_j \) and \( \varepsilon_j \) at equilibrium where \( \dot{\delta}_j = \sigma_j = 0 \). Putting \( \dot{\delta}_j = \sigma_j = 0 \) in Eqs. (2a) and (2b), the values of \( \overline{\xi}_j, \overline{\varepsilon}_j \) are obtained as

\[
\overline{\xi}_j = \frac{\delta}{\alpha_j} \varepsilon \quad \text{and} \quad \overline{\varepsilon}_j = \frac{1}{c_j} \varepsilon.
\]

The partial differential equation governing the motion of an Euler-Bernoulli beam with no distribution of external load is given by

\[
\rho A \dddot{v} = \frac{\sigma_y}{\delta} \int_A \sigma y dA \quad (5)
\]

where, \( \rho \) is the density of the material, \( A \) is the cross-sectional area, \( \sigma \) is the stress in a fibre of infinitesimal cross-section at a distance \( y \) from the neutral axis of the section. The symbols \( (\cdot) \) and \( (\cdot)' \) stand for single partial differentiation with respect to time, \( \partial/\partial t \) and space, \( \partial/\partial x \) respectively.

Substituting the expression of \( \sigma \) from Eqs. (2) into Eq. (5), putting \( \overline{\xi}_j = \frac{\delta}{\alpha_j} \varepsilon \) and \( \overline{\varepsilon}_j = \frac{1}{c_j} \varepsilon \) in Eq. (4) and using \( \varepsilon = \partial u/\partial x = -yv''(x) \), the equations of motion of the beam are obtained by performing the integration over the cross-sectional area and are given as:

For ATF approach as

\[
\rho A \dddot{v} + \sum_{j=1}^{N} E_j I_j v''' = -\sum_{j=1}^{N} \delta_j \int_A \overline{\xi}_j y dA \quad (6a)
\]

\[
\int_A \left[ \dot{\xi}_j + B_j \overline{\xi}_j \right] y dA = -\left( \frac{B_j}{\alpha_j} \right) v'' \quad (6b)
\]

and for ADF approach as
where, \( I \) is the area moment of inertia.

Assuming \( \xi_t(x, y, t) \) and \( \varepsilon_t(x, y, t) \) to vary linearly with the distance of the fibre from the neutral axis, \( y \), i.e., \( \xi_t(x, y, t) = y f_j(x, t) \) and \( \varepsilon_t(x, y, t) = y f_j(x, t) \). Eqs. (6b) and (7b) are rewritten as

\[
\int_{A} [\xi_j^y + b_j \varepsilon_j^y] y dA = -I \left( \frac{b_j}{c_j} \right) v''
\]

In the preceding equations, \( f_j \) stands for dissipation field for the \( j \)th layer due to bending.

Combining Eqs. (6) to (8), the equations of motion for the viscoelastic beam are written

For ATF approach as

\[
\rho A \ddot{v} + \sum_{j=1}^{N} E_j I_j v'''' = -\sum_{j=1}^{N} E_j \int_{A} \xi_j^y y dA
\]

(9a)

\[
\int_{A} \left[ \xi_j^y + B_j f_j + \frac{B_j}{\alpha_j} \right] y dA = \ddot{f}_j
\]

(9b)

and for ADF approach as

\[
\rho A \ddot{v} + \sum_{j=1}^{N} E_j I_j v'''' = -\sum_{j=1}^{N} E_j \int_{A} \xi_j^y y dA
\]

(10a)

\[
\dot{f}_j + B_j f_j + \frac{b_j}{c_j} v'' = 0
\]

(10b)

2.3 Finite element formulation

For the FE formulation of the spatial continuum, the time varying mechanical displacement \( v(x, t) \) and the dissipation coordinate \( f(x, t) \) are approximated by \( v(x, t) = \phi(x)^T q(t) \) and \( f(x, t) = \theta(x)^T p(t) \), where, \( \phi(x) \) is the Hermite shape function and \( \theta(x) \) is the Lagrange shape function. Galerkin's method is employed to the differential equations to derive the finite element form (Lesieutre and Mingori 1990). Fig. 2 illustrates a beam element and the nodal variables for the two dependent fields.

After FE formulation the ATF and ADF approaches yield the following differential equations

\[
([M]_f + [M_R])\{\ddot{q}\} + [K_R]\{q\} = -[J]\{p\}
\]

(11a)

\[
[C]\{\ddot{p}\} + [H]\{p\} = -[F]\{q\}
\]

(11b)
Where, \( M_T = \int_0^l \rho A \phi \phi^T \, dx \), \( M_R = \int_0^l \rho I \phi \phi^T \, dx \) are the mass matrix in translation and the rotary inertia matrix respectively. The rotary inertia matrix has been included to incorporate the effect of the rotation of the cross section of the beam due to bending.

The bending stiffness matrix is given as,

\[
[K_B] = \sum_{j=1}^{N} E_j \int_0^l \phi'' \phi''^T \, dx
\]

The other matrices are

\[
[J] = \begin{bmatrix} [J_1]_{(4x2)} & [J_2]_{(4x2)} & \cdots & [J_N]_{(4x2)} \end{bmatrix}, \quad [C] = \begin{bmatrix} [C_1]_{(2x2)} & 0 & \cdots & 0 \\ 0 & [C_2]_{(2x2)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & [C_N]_{(2x2)} \end{bmatrix}, \quad [F] = \begin{bmatrix} [F_1]_{(2x4)} \\ [F_2]_{(2x4)} \\ \vdots \\ [F_N]_{(2x4)} \end{bmatrix}
\]

\[
[H] = \begin{bmatrix} [H_1]_{(2x2)} & 0 & \cdots & 0 \\ 0 & [H_2]_{(2x2)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & [H_N]_{(2x2)} \end{bmatrix}, \quad [p] = \begin{bmatrix} \{p_1\}_{(2x1)} \\ \{p_2\}_{(2x1)} \\ \vdots \\ \{p_N\}_{(2x1)} \end{bmatrix}
\]

where, \( J_j = \int_0^l \delta_j \phi'' \phi''^T \, dx \), \( C_j = \int_0^l \theta \phi \phi^T \, dx \), \( H_j = \int_0^l B_j \theta \phi \phi^T \, dx \), \( F_j = \int_0^l \left( \frac{B_j \delta}{\alpha_j} \right) \phi'' \phi''^T \, dx \) for ATF approach and \( J_j = \int_0^l E_j \phi'' \phi''^T \, dx \), \( C_j = \int_0^l \theta \phi \phi^T \, dx \), \( H_j = \int_0^l B_j \theta \phi \phi^T \, dx \), \( F_j = \int_0^l \left( \frac{b_j}{c_j} \right) \phi'' \phi''^T \, dx \) for ADF approach, with ‘\( j \)’ varying from 1 to \( N \), the number of layers used in the model.
Eqs. (11a) and (11b) may be further combined and rewritten as:

\[ [A] \{ \ddot{X} \} + [\beta] \{ X \} = \{ P \} \]  
\[ (12) \]

where 
\[ [A] = \begin{bmatrix} [M_d] + [M_a] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [C] \end{bmatrix}, \quad [\beta] = \begin{bmatrix} [0] & [K_a] & [J] \\ [-I] & [0] & [0] \\ [0] & [I] & [H] \end{bmatrix}, \quad \{ X \} = \begin{bmatrix} \{ q \} \\ \{ p \} \end{bmatrix}, \quad \{ P \} = \begin{bmatrix} \{ 0 \} \end{bmatrix} \]

2.4 Modal damping ratio

To obtain the modal damping ratio for multiple viscoelastic layered beam, the complex modulus is derived first. Following Lesieutre (1989) the complex modulus of viscoelastic beam having several layers is obtained by applying Fourier analysis to the equations of motion (Eq. (9) and Eq. (10)). So letting 
\[ v = V e^{i(o \cdot k)} \quad \text{and} \quad f_j = Z_j e^{i(o \cdot k)} \]  
(13a)  
\[ \left( \rho \omega^2 V - \frac{1}{A_j} \sum_{j=1}^{N} E_j k^4 V - \frac{1}{A_j} \sum_{j=1}^{N} \delta_j k^2 Z_j \right) = 0 \]  
(13b)

Substituting \( Z \) from Eq. (13b) to Eq. (13a) yields

\[ -\rho \omega^2 V + \frac{1}{A_j} \sum_{j=1}^{N} E_j k^4 V - \frac{1}{A_j} \sum_{j=1}^{N} \delta_j k^2 Z_j = 0 \]  
(14)

After separating the real and imaginary parts of the preceding equation, the complex modulus is found as

\[ E' (\omega) = \frac{1}{A_j} \sum_{j=1}^{N} E_j \left[ \frac{1}{c_j} + \frac{\omega}{b_j} \right] \left( \frac{1}{c_j} + \frac{\omega}{b_j} \right) \left( \frac{1}{c_j} + \frac{\omega}{b_j} \right) \]  
(15a)

Following the same method, the complex modulus obtained from the ADF approach is given as

\[ E' (\omega) = \frac{1}{A_j} \sum_{j=1}^{N} E_j \left( \frac{1}{c_j} + \frac{\omega}{b_j} \right) \left( \frac{1}{c_j} + \frac{\omega}{b_j} \right) \left( \frac{1}{c_j} + \frac{\omega}{b_j} \right) \]  
(15b)

The modal damping ratio in the ATF approach is given as
3. Finding out the viscoelastic parameters

3.1 Model parameters from hysteresis

This section attempts to find out the values of the viscoelastic parameters \( B, \alpha, \delta \) for ATF approach and \( (b, c) \) in the ADF approach from the values of storage modulus and loss factors usually given in handbooks. Genetic algorithm has been used for finding out those parameters. Viscoelastic parameters have been found out by minimizing the objective function, constructing the error between the hysteresis obtained by the complex modulus approach and the same in the ATF or ADF approach, at any frequency of excitation. A code in MATLAB (version 7.1) has been written for obtaining the viscoelastic parameters of the material.

As an example, a steel rod (length 0.75 m and outer diameter 0.05 m) is considered to be acted upon a sinusoidal axial load \( (P = 1 \text{ N}) \), as shown in the Fig. 3. The material properties of steel (Lazan (1968)) are \( E = 2.1 \times 10^11 \text{ Pa}, \eta = 1.6 \times 10^{-3}, \rho = 7800 \text{ kg/m}^3 \). The equations of motion of uniaxial viscoelastic rod using ATF and ADF approaches have been taken from Lesieutre and Mingori (1990) and Lesieutre et al. (1996) respectively.

Fig. 4 compares the force displacement plots obtained by using ATF, ADF and complex modulus approaches. Response and energy dissipation for various excitation frequencies have been shown in Fig. 5. Close match in all these plots proves the correctness of the viscoelastic parameters.
Fig. 3 Schematic diagram of a bar

Fig. 4 Hysteresis loop

Fig. 5 Response and energy dissipation
Based on the minimization of the error between the hysteresis predicted by the complex modulus approach and the ATF and ADF approaches, values of the viscoelastic parameters are found out. These are $B = 1.807 \times 10^5$, $\alpha = 1.807 \times 10^5$, $\delta = 3.186 \times 10^7$ and $b = 1.807 \times 10^5$, $c = 37.3748$.

3.2 ATF parameters from ADF parameters

As a part of this extraction process it was felt necessary to set up relationships among the model parameters in different approaches so that knowing model parameters in one approach enables one to calculate the parameters in another approach. Yet again it may be seen that extracting the ADF parameters is easier and less time consuming than the ATF parameters as number and the range of values of ADF parameters are lesser than those of the ATF parameters. Comparing the expressions of complex modulii for ATF and ADF approaches as given in Eqs. (15a) and (15b), the ATF parameters in terms of the ADF parameters for $j^{th}$ layer are found out as given below.

$$B_j = h_j, \quad \alpha_j = B_j, \quad \delta_j = \frac{E_j b_j}{\eta_j c_j}$$

4. Forced response of a beam

The force vector $\{ \Phi \}$ in Eq. (12) is written as $\Phi = \bar{\Phi} e^{i\omega t}$. The vector $\{ \bar{\Phi} \}$ contains amplitude of excitation vectors at the excitation nodes.

Substituting a trial solution $\{ X \} = \{ \bar{X} \} e^{i\omega t}$ in Eq. (12) yields

$$\{ \bar{X} \} = ([\theta] + i\omega[A])^{-1}\{ \Phi \} = [\beta(\omega)]\{ \Phi \}$$

where $[\beta(\omega)]$ in the preceding equation is the frequency response function (FRF) matrix.

The response amplitude at the $n^{th}$ node is given by $X_n = \max(\text{Real}(X_n e^{i\omega t}))$ where, Real() stands for the real part of the complex number.

5. Results and discussions

5.1 Response amplitude and modal damping ratio of a viscoelastic beam

A cantilever beam made of steel (Length $L = 0.75$ m and outer diameter $d = 0.05$ m) as shown in Fig. 6 is considered. The material properties of steel are $E = 200$ GPa, $\eta = 1.6 \times 10^3$ $\rho = 7800$ kg/m$^3$ and the viscoelastic parameters of steel using ATF and ADF approach are $B = 1.807 \times 10^5$, $\alpha = 1.807 \times 10^5$, $\delta = 3.186 \times 10^7$ and $b = 1.807 \times 10^5$, $c = 37.3748$ as given in section 3. The beam is subjected to harmonic excitation at the tip. The amplitude of transverse load ($P$) is taken as 1 N.

Table 1 shows the nice agreement of three consecutive modal damping ratios and natural frequencies by using ATF and ADF approaches. The damping ratios have been found out from the complex modulus.

Fig. 7 shows the response amplitude at the tip for different excitation frequencies. The curve has two peaks which indicate the first and second natural frequencies. The plot shows that the ATF and ADF formulations predict the same response amplitude.
5.2 Beams of multiple viscoelastic layers

The equations of motion of beams composed of multiple viscoelastic layers have been discussed in section 2.3. In this section a specific example of a beam using two viscoelastic layers (Fig. 8) has been considered. Fig. 8 shows the beam of two concentric sections of different materials of densities \( \rho_i \) and \( \rho_o \). The radius ratio of the section and equivalent density are defined as

\[
\mathcal{R} = \frac{r_i}{r_o}, \quad \rho = \rho_i \mathcal{R}^2 + \rho_o (1 - \mathcal{R}^2)
\]

If the inner and outer materials are interchanged keeping the masses of the inner and outer material unchanged the radius ratio becomes

Table 1 Damping ratio and natural frequency for three consecutive modes

<table>
<thead>
<tr>
<th>Method</th>
<th>1st mode</th>
<th>2nd mode</th>
<th>3rd mode</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Damping ratio</td>
<td>Natural frequency (Hz)</td>
<td>Damping ratio</td>
</tr>
<tr>
<td>ATF</td>
<td>3.0397e-5</td>
<td>63.6141</td>
<td>1.8974e-4</td>
</tr>
<tr>
<td>ADF</td>
<td>3.0405e-5</td>
<td>63.6139</td>
<td>1.8978e-4</td>
</tr>
</tbody>
</table>

Fig. 7 Beam Response amplitude
As \( R \to 0 \) the whole cross section is filled by the outer material of density \( \rho_o \), and \( R \to 1 \) when the cross section is made of the inner material of density of \( \rho_i \).

For an example the same beam as in section 5.1 (Length \( L = 0.75 \) m and outer diameter \( D = 0.05 \) m) is considered to have two materials (similar to the one shown in Fig. 8), aluminium and steel, one forming the core and the other the outer layer. Following Lesieutre and Mingori (1990) the material properties of aluminium are taken as \( E = 7.13 \times 10^9 \) Pa, \( \rho = 2750 \) kg/m\(^3\) and the viscoelastic parameters in ATF approach are as \( B = 8000 \), \( \alpha = B \), \( \delta = 4.7766 \times 10^6 \). The corresponding properties of steel are taken as \( E = 2.1 \times 10^{11} \) Pa, \( \rho = 7800 \) kg/m\(^3\), \( B = 1.807 \times 10^5 \), \( \alpha = 1.807 \times 10^5 \), \( \delta = 3.186 \times 10^7 \), as given in section 3.

5.2.1 Modal damping ratio and change in the First Natural Frequency in bending

Fig. 9 shows the modal damping ratio (\( \zeta \)) for various radius ratios (\( R \)). The study has been carried out for both placements of aluminium and steel, i.e., steel as core and aluminium forming the outer layer and vice versa. When steel forms the core and aluminium forms the outer layer (Case-I), \( R \) is defined as \( R = r_i/r_o = r_{steel}/r_o \). It is seen from the figure that, for this case (as shown by the dotted line) \( \zeta \) decreases with increasing values of \( R \) as steel provides lesser damping than aluminium. The solid lines show the corresponding values of \( \zeta \), when for the same beam mass aluminium forms the core and steel forms the outer layer (Case-II), and the radius ratio is \( R' = \sqrt{1-R^2} \) as shown in Eq. (19). For this placement the values of modal damping ratio fall because, firstly aluminium being placed as the core material is not subjected to flexural rate as high as the former situation (Case-I) and secondly the placement of steel on the outer layer enhances modal stiffness as it undergoes higher deformation due to bending in comparison with the earlier situation (Case-I).

Fig. 10 shows the % change in First Natural Frequency (FNF) with respect to the case of an all aluminium beam for various values of \( R \). The dotted line is for Case-I whereas the solid line represents the results corresponding to Case-II. In consonance with the observation held in explaining Fig. 9 the FNF in Case-II (for which the change is always positive) is found to be more than that for Case-I (for which the change is always negative) always. From the figure it is seen that there exists an optimum value of \( R \) beyond which the FNF in Case-II starts decreasing. This is due to the fact that the equivalent density increases with \( R \), as specific gravity of steel is more than that of aluminium. So a critical value of \( R \) is reached when aluminium forms the core and steel the outer layer. For an example the value of \( R \) corresponding to the maximum increment of FNF (15%) for Case-II is found to be 0.6 which corresponds to a value of \( R' = 0.8 \). For the dimensions chosen,
The beam is composed of an aluminium core of diameter 0.04 m with an outside steel layer of thickness 0.005 m. Thus with proper dimension and placement of materials the natural frequencies may be substantially increased. This is the utility of using a composite beam.

5.2.2 Response due to harmonic excitation
The beam is subjected to transverse harmonic excitation of amplitude 1 N at the tip. Fig. 11 shows the response amplitude plotted at several excitation frequencies for different values of $R$ as well as different placements (inside and outside) of aluminium and steel. For Case-I the resonance peaks occur earlier as $R$ increases; this is in keeping with the findings in Fig. 10 where FNF is seen to fall initially as $R$ increases as the mass of the beam increases more than its stiffness as the radius of steel core increases. For Case-II however the peak shifts to the right as $R$ increases; this behaviour may similarly be explained from Fig. 10. As the resonance gets postponed, the response
amplitude below the resonant frequency keeps falling as $R$ increases. For an example a value of $R = 0.5$ in the present configuration (Length $L = 0.75$ m and outer diameter $D = 0.05$ m) will give a core radius of steel = 0.0125 m with an aluminium outer layer of 0.0125 m. A beam same as mass of this beam will be composed of an aluminium core of radius 0.0217 m and a steel outer layer of thickness 0.0033 m corresponding to a value of $R' = 0.866$ as in Case-II.

Fig. 12(a) shows the amplitude of stress distribution of a composite beam with an aluminium core
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of diameter 0.04 m and steel layer of thickness 0.005 m (Ψ' = 0.8) at sections \( L' = 0.1 \) and 0.5 where \( L' = l/L \) and \( r' = r/r_0 \), \( 'r' \) being the distance of the beam section from the root, \( r' \) being the radial location of any point and \( r_0 = 0.025 \) m is the outer radius of the beam cross-section. It is seen that as the point moves from Aluminium domain to the steel domain, the stress increases suddenly as steel has higher modulus of elasticity. Similarly the Fig. 12(b) shows the variation of stress amplitude as the section is shifted along the length of the cantilever from the fixed to the free end. The stress amplitude is seen to fall as expected from root to the tip however the stress magnitude in steel always remains more than that of Aluminium.

6. Conclusions

1. The ATF & ADF approach introduce internal variables which are very efficient in modelling frequency dependent viscoelastic material properties in the time domain.
2. It has been noticed that the ADF approach is more convenient to apply than the ATF approach as the number of parameters and their range of values in ADF are less than those in ATF approach.
3. By proper placement of the composite layers and proper selection of thickness ratios, the first natural frequency may be sufficiently postponed leading to reductions in vibration response.

References

Notation

\( A \): cross-sectional area of the beam
\( \mathcal{A} \): affinity, a thermodynamically conjugate to \( \xi \)
\( b, B \): inverse of relaxation time
\( E \): unrelaxed elastic modulus
\( E^s \): anelastic material property
\( f \): ATF or ADF for beam bending, corresponding to \( \xi \) or \( \epsilon^i \)
\( \mathcal{H} \): Helmholtz free energy density function
\( I \): area moment of inertia
\( l \): element length
\( L \): length of the beam
\( L^* \): non-dimensional distance from the root
\( p \): vector of ADF or ATF displacements
\( \Phi \): amplitude of harmonic excitation
\( q \): vector of mechanical displacements
\( r_i \): radius occupied by the inner material
\( r_o \): outer radius of the beam
\( r^* \): non-dimensional distance from the neutral surface
\( \mathcal{R} \): radius ratio, \( \mathcal{R} = r/r_o \)
\( t \): time in sec
\( u, v \): mechanical displacement corresponding to the \( x \) and \( y \) axis respectively
\( \alpha \): material property relating changes in \( \mathcal{A} \) to those in \( \epsilon^i \)
\( \beta \): frequency response function (FRF) matrix
\( \delta \): strength of coupling between the mechanical displacement field and the thermodynamic field
\( \epsilon \): mechanical strain
\( \epsilon^i \): anelastic strain
\( \phi \): Hermite shape function
\( \Theta \): Lagrange shape function
\( \rho \): equivalent density of the composite material i.e., layered beam
\( \sigma \): mechanical stress
\( \sigma^i \): anelastic stress
\( \xi \): modal damping ratio
\( \omega \): excitation frequency
\( \omega_h \): modal frequency
\( \zeta \): ATF displacement