TRANSIENT ANALYSIS OF ERROR SATURATION NONLINEARITY LMS IN IMPULSIVE NOISE

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ABSTRACT

This paper introduces a new approach for the transient analysis of adaptive filter with error saturation nonlinearity in presence of impulsive noise. The recursive expressions for mean-square deviation(MSD) and excess mean-square error(EMSE) are derived based on weighted energy conservation arguments. The approach is leads to new performance results without restricting the input regression data to be white.

1. INTRODUCTION

It is known that when data is contaminated with non-Gaussian noise, the linear systems provides poor performance. In many physical environment the additive noise is modeled as impulsive and is characterized by long-tailed non-Gaussian distribution. The performance of the system is evaluated under the assumption that the Gaussian noise is severally degraded by the non-Gaussian or Gaussian mixture [1] noise due to deviation from normality in the tails [2, 3]. The effects of saturation type of non-linearity on the least-mean square adaptation for Gaussian inputs and Gaussian noise have been studied [4, 5]. Recent research focus is to develop adaptive algorithm that are robust to impulsive noise or outlier present in the training data. Number of algorithms have been proposed [3,6–8] to reduces the effects of impulsive noise. This class of algorithms is difficult to analyze and therefore it is not uncommon to resort to different methods and assumptions. Prof J. Bershad in his recent paper [9] has showed that the error saturation nonlinearities LMS provides good performance in presence of impulsive noise. How ever he has not given the recursion relation for correlated input data.

The least-mean square(LMS) algorithm is popular adaptive algorithm because of its simplicity [10], [11]. Many LMS type algorithms have been suggested and analyzed in literature is the class of least-mean square algorithm with error saturation nonlinearity is of particular importance. The general way of convergence analysis of any type of adaptive algorithms using weight-energy relation is dealt in [12]. Further in some literature the error nonlinearity analysis [13, 14] and data nonlinearity analysis [15] are have been made weighted-energy conservation method. The theory dealt in [9] provides the idea of the subsequent analysis of Gaussian mixture case. It also suggests how it can applied to each component separately to obtain recursive relation for the nonlinear LMS.

In this paper we use both the ideas to develop a new generalized method to obtain the transient analysis of saturation nonlinearity LMS in presence of Gaussian contaminated impulsive noise. We have derived the performance equations by assuming that the input data is Gaussian uncorrelated. This idea can also extended to the case of correlated input regressor data. Finally it shown that the theoretical performance curves have excellent agreement with the corresponding simulation results.

2. ADAPTIVE ALGORITHM WITH SATURATION ERROR NONLINEARITY

The estimate of an $M \times 1$ unknown vector $\mathbf{w}^{\circ}$ by using row regressor $\mathbf{u}_i$, of length M and output samples $d(i)$ that is given as

$$d(i) = \mathbf{u}_i \mathbf{w}^{\circ} + v(i) \tag{1}$$

Where $v(i)$ is represents the impulsive noise instead of Gaussian. Out of many adaptive algorithms proposed in literature [10, 11] the well known LMS algorithm is analyzed. Its weight update equation is given by

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu e(i) \mathbf{u}_i^T$$

In this paper we focus on a slightly different class of algorithm by introducing an error nonlinearity into the feedback error signal so that the weight update equation can be written as

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \mathbf{u}_i^T f[e(i)] \quad i \geq 0 \tag{2}$$

where $\mathbf{w}_i$ is the estimate of $\mathbf{w}$ at time i and μ is the step size

$$e(i) = d(i) - \mathbf{u}_i \mathbf{w}_{i-1} = \mathbf{u}_i \mathbf{w}^{\circ} - \mathbf{u}_i \mathbf{w}_{i-1} + v(i) \tag{3}$$

and

$$f(y) = \int_0^y \exp[-u^2/2\sigma_s^2] du = \sqrt{\frac{\pi}{2}} erf\left[\frac{y}{\sqrt{2}\sigma_s}\right] \tag{4}$$

where σ_s is a parameter that defines the degree of saturation.

If $f(e)$ represents the cost function, then its gradient is defined as

$$\begin{aligned} \frac{\partial f}{\partial \mathbf{w}} &= \frac{\partial f}{\partial e} \cdot \frac{\partial e}{\partial \mathbf{w}} \\ &= \frac{\partial f}{\partial e} \cdot (-\mathbf{u}) \end{aligned}$$

If we choose the cost function $f(e) = e^2$ then $\frac{\partial f}{\partial e} = 2e$ is linear, otherwise $\frac{\partial f}{\partial e}$ is a non-linear function of e. In this approach nonlinear $f(e)$ and is so chosen that of $\frac{\partial f}{\partial e}$ is also nonlinear *i. e.*.

$$f(e) = \int \left(\frac{\partial f}{\partial e}\right) de$$

In this saturation non-linearity LMS case we have chosen Gaussian nonlinearity on error.

2.1 Model for Impulsive Noise

The transient analysis of adaptive filters that are available in literature is for white Gaussian noise case. But in real environment impulsive noise is encountered. The impulsive noise is modeled as a two component of the Gaussian mixture [1, 3] which is given by

$$n_o(i) = n_g(i) + n_{im}(i) = n_g(i) + b(i)n_w(i) \tag{5}$$

where $n_g(i)$ and $n_w(i)$ are independent zero mean Gaussian noise with variances σ_g^2 and σ_w^2, respectively; $b(i)$ is a switch sequence of ones and zeros, which is modeled as an iid Bernoulli random process with occurrence probability $P_r(b(i)=1) = p_r$ and $P_r(b(i) = 0) = 1 - p_r$. The variance of $\eta_w(i)$ is chosen to be very large than that of $n_g(i)$ so that when $b(i) = 1$, a large impulse is experienced in $n_o(i)$. The corresponding pdf of $n_o(i)$ in (5) is given by

$$f_{n_o}(x) = \frac{1-p_r}{\sqrt{2\pi}\sigma_g}\exp\left(\frac{-x^2}{2\sigma_g^2}\right) + \frac{p_r}{\sqrt{2\pi}\sigma_\Sigma}\exp\left(\frac{-x^2}{2\sigma_\Sigma^2}\right) \tag{6}$$

where $\sigma_\Sigma^2 = \sigma_g^2 + \sigma_w^2$ and $E[n_o^2(i)] = \sigma_g^2 + p_r\sigma_w^2$. It is noted that when $p_r = 0$ or 1, $n_o(i)]$ is a zero-mean Gaussian variate.

3. TRANSIENT ANALYSIS

We are interested in studying the time-evolution of the variances $E|e(i)|^2$ and $E\|\tilde{\mathbf{w}}_i\|^2$ where

$$\tilde{\mathbf{w}} = \mathbf{w}^\circ - \mathbf{w}_i$$

The steady-state values of the variances known as mean-square error and mean-square deviation performance of the filter. In order to study the time evaluation of above variances, we introduce [12] the weighted *a priori* and *a posteriori* error defined as

$$e_a^\Sigma(i) = \mathbf{u}_i\Sigma\tilde{\mathbf{w}}_{i-1} \qquad \text{and} \qquad e_p^\Sigma(i) = \mathbf{u}_i\Sigma\tilde{\mathbf{w}}_i \tag{7}$$

where Σ is a symmetric positive definite weighting matrix. It will be seen that the different choice for Σ allows us to evaluate different performance. For $\Sigma = I$ we define

$$e_a(i) = e_a^I(i) = \mathbf{u}_i\tilde{\mathbf{w}}_{i-1}, \qquad e_p(i) = e_p^I(i) = \mathbf{u}_i\tilde{\mathbf{w}}_i \tag{8}$$

Subtracting $\mathbf{w}^\circ$ from both sides of (2), we get

$$\tilde{\mathbf{w}}_i = \tilde{\mathbf{w}}_{i-1} - \mu f[e(i)]\mathbf{u}_i \tag{9}$$

Using the definition of a priori error in (3), we get

$$e(i) = e_a(i) + v(i) \tag{10}$$

Relation between various error terms $e_a^\Sigma(i)$, $e_p^\Sigma(i)$ and $e(i)$ is obtained by premultiplying both sides of (9) by $\mathbf{u}_i\Sigma$

$$e_p^\Sigma(i) = e_a^\Sigma(i) - \mu f[e(i)]\|\mathbf{u}_i\|_\Sigma^2 \tag{11}$$

3.1 Weight-Energy Relation

Elimination of the nonlinearity $f[e(i)]$ from (9) by using (11) we obtained

$$\tilde{\mathbf{w}}_i = \tilde{\mathbf{w}}_{i-1} + \frac{e_p^\Sigma(i) - e_a^\Sigma(i)}{\|\mathbf{u}_i\|^2}\mathbf{u}_i^T \tag{12}$$

Taking weighted energy on both sides of (12), we get

$$\|\tilde{\mathbf{w}}_i\|_\Sigma^2 + \frac{|e_a^\Sigma(i)|^2}{\|\mathbf{u}_i\|_\Sigma^2} = \|\tilde{\mathbf{w}}_{i-1}\|_\Sigma^2 + \frac{|e_p^\Sigma(i)|^2}{\|\mathbf{u}_i\|_\Sigma^2} \tag{13}$$

The variance relation can be obtained from the energy relation (13) by replacing a posteriori error by its equivalent expression.

$$\|\tilde{\mathbf{w}}_i\|_\Sigma^2 = \|\tilde{\mathbf{w}}_{i-1}\|_\Sigma^2 - 2\mu e_a^\Sigma(i)f[e(i)] + \mu^2\|\mathbf{u}_i\|_\Sigma^2 f^2[e(i)]$$

Taking expectation on both sides we get the same equation as in [13] which is given as:

$$\begin{aligned} E[\|\tilde{\mathbf{w}}_i\|_\Sigma^2] = E[\|\tilde{\mathbf{w}}_{i-1}\|_\Sigma^2] &- 2\mu E[e_a^\Sigma(i)f[e(i)]] \\ &+ \mu^2 E[\|\mathbf{u}_i\|_\Sigma^2 f^2[e(i)]] \end{aligned} \tag{14}$$

Evaluation of 2nd and 3rd terms on RHS of (14)is difficult as it contains the nonlinearity term. To evaluate the transient analysis we make the same assumption taken in [13].

- The noise sequence $v(i)$ is iid and independence of $\mathbf{u}_i$
- For any constant matrix Σ and for all i, $e_a(i)$ and $e_a^\Sigma(i)$ are jointly Gaussian.
- The adaptive filter is long enough such that the weighted norm of input regressor and the square of error nonlinearity i.e. $f^2[e(i)]$ are uncorrelated.

Price's theorem [16, 17]plays an important rule to analyze the 2nd term on RHS of equation(14) which is given as

$$E[xf[y+z]] = \frac{E[xy]}{E[y^2]}E[yf[y+z]]$$

where x and y be jointly Gaussian random variables that are independent from the third random variable z. Here the third term is given as independent of x and y which are jointly Gaussian. In [9], [13] the noise is considered as simply Gaussian and independent of the errors $e_a(i)$ and $e_a^\Sigma(i)$. But in this paper we consider the noise is impulsive and also assume that this impulsive noise also independent of errors. So by using the the Price's theorem and assuming that the impulsive noise is independent of errors we get the same general equation [17, [13]] which is given as

$$E[e_a^\Sigma(i)f[e(i)]] = E[e_a^\Sigma(i)e(i)]h_G[E[e_a^2(i)] \tag{15}$$

where the general expression for h_G is given as

$$h_G = \frac{\sigma_s}{\sqrt{E[e_a^2(i)] + \sigma_s^2}}E\left[\exp\left[-\frac{v^2(i)}{2(E[e_a^2(i)] + \sigma_s^2)}\right]\right]$$

Now we can evaluate the value of h_G using the pdf $p_v(v)$ in (6) as

$$h_G = \frac{(1-p_r)\sigma_s}{\sqrt{E[e_a^2(i)] + \sigma_s^2 + \sigma_g^2}} + \frac{p_r\sigma_s}{\sqrt{E[e_a^2(i)] + \sigma_s^2 + \sigma_\Sigma^2}} \tag{16}$$

In similar way we can evaluate the third term of (14) by taking long filter assumption for which the weighted norm of input data and the squared error nonlinearity are uncorrelated as in [13]. But here the expression for $h_U = E[f^2[e(i)]]$ is evaluated by assuming the noise is impulsive and whose pdf is given in (6). The expression of h_U is given in (17).

$$\begin{aligned} h_U = (1-p_r)\sigma_s^2\sin^{-1}&\left(\frac{\sigma_g^2 + E[e_a^2(i)]}{E[e_a^2(i)] + \sigma_s^2 + \sigma_g^2]}\right) \\ &+ p_r\sigma_s^2\sin^{-1}\left(\frac{\sigma_\Sigma^2 + E[e_a^2(i)]}{E[e_a^2(i)] + \sigma_s^2 + \sigma_\Sigma^2}\right) \end{aligned} \tag{17}$$

3.2 Weighted-Energy Recursion Relation

Employing the same assumption as in [13] and assuming the sequence $\mathbf{u}_i$ is zero-mean iid, and has covariance matrix $\mathbf{R}$, the weighted-energy recursion relation is given as

$$\begin{aligned} E[\|\tilde{\mathbf{w}}_i\|_\Sigma^2] = E[\|\tilde{\mathbf{w}}_{i-1}\|_\Sigma^2] &- 2\mu h_G E[\|\tilde{\mathbf{w}}_{i-1}\|_{\Sigma\mathbf{R}}^2] \\ &+ \mu^2 E[\|\mathbf{u}_i\|_\Sigma^2]h_U \end{aligned} \tag{18}$$

4. RECURSION EQUATIONS

The learning curve of the filters refers to the time-evolution of the variances $E[e_a^2(i)]$ and $E[\|\tilde{\mathbf{w}}_i\|^2]$. the steady-state values are called as excess-mean square error(EMSE) and mean-square deviation(MSD) respectivelly. Under independent assumption we can write the variance $E[e^2(i)]$ as

$$E[e^2(i)] = E[\|\tilde{\mathbf{w}}_{(i-1)}\|^2_{\mathbf{R}}]$$

This suggests that the learning curve can be evaluated by computing the weight-energy relation (18) for each i and by choosing the weight parameter $\Sigma = \mathbf{R}$ for EMSE and $\Sigma = \mathbf{I}$ for MSD respectively. Here we develop the recursive relation for EMSE and MSD first for white input data and then extend to correlated input data.

4.1 Case of White Regressor Data

In case of white input regressor data, the covariance matrix $\mathbf{R} = \sigma_u^2\mathbf{I}$, so that $E[e_a^2(i)] = \sigma_u^2 E[\|\tilde{\mathbf{w}}_{(i-1)}\|^2]$. Therefore the (18) can be solved as

$$E[\|\tilde{\mathbf{w}}_i\|^2_\Sigma] = E[\|\tilde{\mathbf{w}}_{i-1}\|^2_\Sigma] - 2\mu h_G \sigma_u^2 E[\|\tilde{\mathbf{w}}_{i-1}\|^2_\Sigma] + \mu^2 E[\|\mathbf{u}_i\|^2_\Sigma] h_U \tag{19}$$

Thus, setting $\Sigma = \mathbf{I}$ in the above equation for MSD recursion equation, we get

$$E[\|\tilde{\mathbf{w}}_i\|^2] = E[\|\tilde{\mathbf{w}}_{i-1}\|^2] - 2\mu h_G \sigma_u^2 E[\|\tilde{\mathbf{w}}_{i-1}\|^2] + \mu^2 E[\|\mathbf{u}_i\|^2] h_U \tag{20}$$

Substituting $\eta(i) = E[\|\tilde{\mathbf{w}}_i\|^2]$, (20) written as

$$\eta(i) = \eta(i-1) - 2\mu h_G \sigma_u^2 \eta(i-1) + \mu^2 M \sigma_u^2] h_U \tag{21}$$

where the nonlinear parameter h_G and h_U are given as below:

$$h_G = \frac{(1-p_r)\sigma_s}{\sqrt{\sigma_u^2\eta(i-1) + \sigma_s^2 + \sigma_g^2}} + \frac{p_r\sigma_s}{\sqrt{\sigma_u^2\eta(i-1) + \sigma_s^2 + \sigma_\Sigma^2}} \tag{22}$$

$$h_U = (1-p_r)\sigma_s^2 \sin^{-1}\left(\frac{\sigma_g^2 + \sigma_u^2\eta(i-1)}{\sigma_u^2\eta(i-1) + \sigma_s^2 + \sigma_g^2]}\right) + p_r\sigma_s^2 \sin^{-1}\left(\frac{\sigma_\Sigma^2 + \sigma_u^2\eta(i-1)}{\sigma_u^2\eta(i-1) + \sigma_s^2 + \sigma_\Sigma^2]}\right) \tag{23}$$

Equation (21) shows recursive equation for MSD in case of white input regressor data. This expression is similar to the (26) of [9] except only one extra term in later. In our analysis we have assumed that for long filter the weighted norm of input data $\|\mathbf{u}_i\|^2_\Sigma$ and the error nonlinearity square $f[e^2(i)]$ are uncorrelated. Therefore the extra term in (26) of [9] is not appeared in (21). In addition this extra term can be neglected for small step size.

In the same way we can obtain the recursion equation for EMSE by simply choosing $\Sigma = \mathbf{R}$ in (18) where the time evolution EMSE at i can be written as $\zeta(i) = E[\|\tilde{\mathbf{w}}_i\|^2_{\mathbf{R}}]$.

$$\zeta(i) = \zeta(i-1) - 2\mu h_G \sigma_u^2 \zeta(i-1) + \mu^2 M \sigma_u^4 h_U \tag{24}$$

The nonlinearity parameters h_G and h_U in EMSE of (24) are as

$$h_G = \frac{(1-p_r)\sigma_s}{\sqrt{\zeta(i-1) + \sigma_s^2 + \sigma_g^2}} + \frac{p_r\sigma_s}{\sqrt{\zeta(i-1) + \sigma_s^2 + \sigma_\Sigma^2}} \tag{25}$$

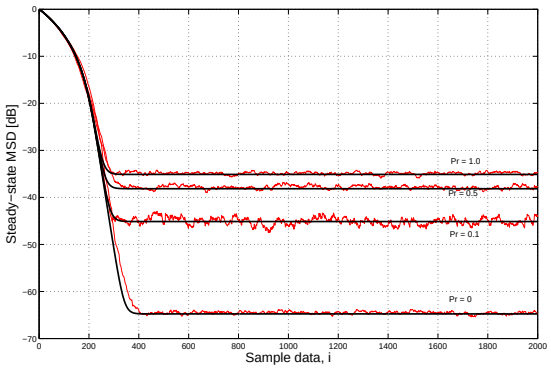

Figure 1: Theoretical (black) and simulated (red) mean-square deviation (MSD) curve for $p_r = 0.0$ (no impulsive noise) 0.1, 0.5, and 1.0

$$h_U = (1-p_r)\sigma_s^2 \sin^{-1}\left(\frac{\sigma_g^2 + \zeta(i-1)}{\zeta(i-1) + \sigma_s^2 + \sigma_g^2]}\right) + p_r\sigma_s^2 \sin^{-1}\left(\frac{\sigma_\Sigma^2 + \zeta(i-1)}{\zeta(i-1) + \sigma_s^2 + \sigma_\Sigma^2]}\right) \tag{26}$$

4.2 Case of Correlated Regressor Data

The results (18) allows us to evaluate the time evolution of the variances without the whiteness assumption on the input regression data i.e. for general matrices $\mathbf{R}$. The main idea is to take the advantage of the free choice of weighted matrix Σ. If we choose $\Sigma = \mathbf{I}, \mathbf{R}, ...\mathbf{R}^{M-1}$, then h_G and h_U remain the same, so that these parameters are independent of choice of weighted matrix Σ.

5. RESULTS

All simulations are carried out using regressors with shift invariance structure to cope with realistic scenario. Therefore the regressor are filled up as

$$\mathbf{u}_i = [u(i), u(i-1), \ldots, u(i-M+1)]^T \tag{27}$$

The recursive equations are derived by assuming that input data are uncorrelated, so that the Monte Carlo simulation technique is used to get simulated value of MSE and EMSE in presence of different percentage of impulsive noise. The desired data are generated according to the model given in (1), and the unknown vector $\mathbf{w}^\circ$ is set to $[1,1,\ldots,1]^T/\sqrt{M}$. Here the back ground noise is Gaussian contaminated impulsive type which is defined in (5). The back ground noise is Gaussian with variance σ_g^2 and the impulsive noise is also Gaussian but it occur with some probability having high variance σ_w^2. The performance of the saturation nonlinearity algorithm in presence of impulsive noise with different percentage is depicted in Figs. 1 and 2. The parameters are chosen as $\mu = 0.05, \sigma_u^2 = 1, \sigma_{sat}^2 = 0.01, \sigma_g^2 = 10^{-6}, \sigma_w^2 = 10^3\sigma_g^2$ which are nearly same as taken in [9]. These figures exhibit excellent match between theoretical and simulation results.

Further we verify that the theoretical and simulation results when the impulsive noise whose variance is much much more than the background noise. The parameters are taken as $\mu = 0.05, \sigma_u^2 = 1, \sigma_{sat}^2 = 0.01, \sigma_g^2 = 10^{-3}, \sigma_w^2 = 10^4\sigma_g^2$. The performance curves are depicted in Figs. 3 and 4. Both the results also shows excellent match.

6. CONCLUSION

In this paper we have used energy-weighted conservation arguments to study the performance of saturation nonlinearity LMS with im-

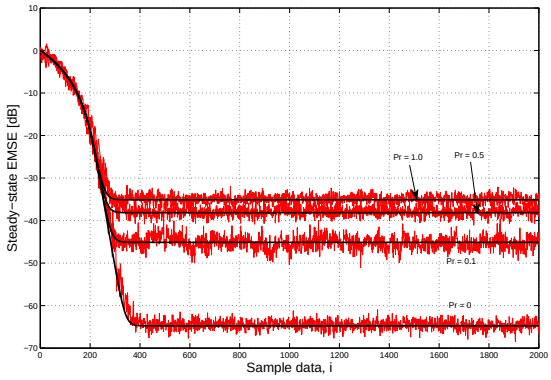

Figure 2: Theoretical (black) and simulated (red) excess-mean-square error (EMSE) curve for $p_r = 0.0$(no impulsive noise) 0.1, 0.5, and 1.0

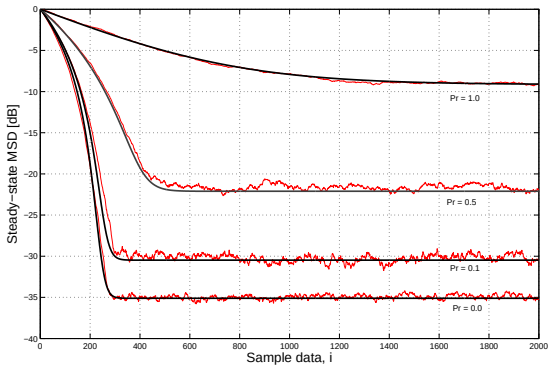

Figure 3: Theoretical (black) and simulated (red) mean-square deviation (MSD) curve for $p_r = 0.0$(no impulsive noise) 0.1, 0.5, and 1.0

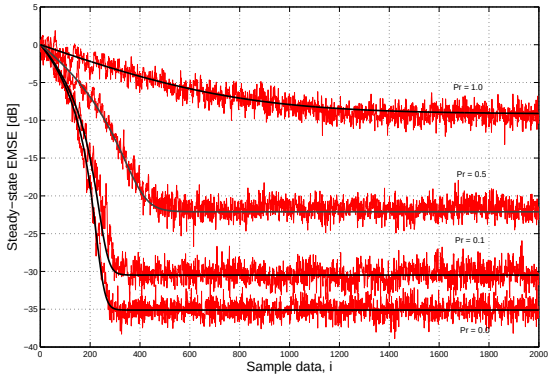

Figure 4: Theoretical (black) and simulated (red) excess-mean-square error (EMSE) curve for $p_r = 0.0$(no impulsive noise) 0.1, 0.5, and 1.0

pulsive noise. The recursion equations for MSD and EMSE are derived in presence of impulsive noise. The simulated results have good agreement with theoretical counter part. We can extend this approach to other family of error nonlinearities like LMF, Sign error *etc.*. Finally this approach can also be applied to general Gaussian mixture type of noise which is more frequently used in RADAR signal processing.

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